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EUROPEAN OPTION PRICING WITH GENERAL TRANSACTION COSTS AND SHORT-SELLING CONSTRAINTS

Ajay Subramanian

Susquehanna Partners, G.P., 401 City Avenue, Suite 220,
Bala Cynwyd, PA 19004, USA

ABSTRACT

In this paper, we study the problem of European Option Pricing in a market with short-selling constraints and transaction costs having a very general form. We consider two types of proportional costs and a strictly positive fixed cost. We study the problem within the framework of the theory of stochastic impulse control. We show that determining the price of a European option involves calculating the value functions of two stochastic impulse control problems. We obtain explicit expressions for the quasi-variational inequalities satisfied by the value functions and derive the solution in the case where the parameters of the price processes are constants and the investor's utility function is linear. We use this result to obtain a price for a call option on the stock and prove that this price is a nontrivial lower bound on the *hedging price* of the call option in the presence of general transaction costs and short-selling constraints. We then consider the situation where the investor's utility function has a general form and characterize the value function as the pointwise limit of an increasing sequence of solutions to associated optimal stopping problems. We thereby devise a numerical procedure to calculate the option price in this general setting and implement the procedure to calculate the option price for the class of exponential utility functions. Finally, we carry out a qualitative investigation of the option prices for exponential and linear-power utility functions.

1. INTRODUCTION

Option pricing has been extensively studied in the mathematical finance literature since the publication of the Black-Scholes formula in 1973. The analysis of Black and Scholes however assumes a perfect market with frictionless trading. These assumptions are clearly idealizations. More recently, several authors have studied the problem of option pricing in the presence of market imperfections such as transaction costs and trading restrictions.The vast literature on the subject can be broadly classified as follows.

There is one group of papers (e.g. Leland [18], Boyle and Vorst[5]) that deals with the problem of hedging call and put options in the presence of *proportional transaction costs*, but the strategies considered in these models are not chosen to satisfy some optimality criteria that investors may wish to meet. The optimality criterion for this problem can be defined in terms of expected utility, i.e. the chosen hedging strategy should maximize the expected utility of the difference between the realized cash flow and the desired one at maturity. These so-called *preference-based* models have been discussed extensively in Hodges and Neuberger [13], Dumas and Luciano [9], Davis and Norman [7], Davis, Panas and Zariphopoulou [8], Magill and Constantinides [19] and the references cited therein. The above papers deal with the problem of option pricing in the presence of *proportional transaction costs*, but in the *absence of trading restrictions* such as *short-selling constraints*.

Another group of papers deals with the so-called *minimum cost of super-replication* approach where the criterion is to minimize the cost of super-replicating the contingent claim in the presence of transaction costs and trading restrictions. As is shown in Section 3 of this paper this approach is, in fact, equivalent to the preference-based approach if the utility function is chosen to have a specific form. Some of the papers in this category are those by Dumas and Luciano [9], Avellaneda and Paras [1], Jouini and Kallal [14], Cvitanic and Karatzas [6], Whalley and Wilmott [23], Edirisinghe, Naik and Uppal [11], among others. The paper by Edirisinghe et al. [11] considers the problem of option pricing in the presence of *proportional and fixed transaction costs and trading restrictions*, i.e. the problem considered in the present paper, but the setting is entirely *discrete*, i.e. there is a pre-specified set of trading dates where the investor can trade and the underlying stock price process is assumed to be a binomial random walk.

There is another group of papers that have considered the problem of *portfolio optimization* in the presence of *fixed transaction costs*. Some of the relevant papers in this area are Eastham and Hastings [10], Hastings [12], Mohamed [20], Korn [16], Bielecki and Pliska [4] and the references cited therein.

The recent book by Karatzas and Shreve [15] contains a detailed exposition of many of the above topics and also provides an extensive list of references to the substantial literature on portfolio optimization and option pricing in the presence of market imperfections.

The present paper extends the existing literature on option pricing in the presence of market imperfections in several ways. It considers, for the first time, the problem of option pricing in the presence of general transaction costs, i.e. proportional as well as fixed costs and short-selling constraints in a general continuous-time setting. This problem is very important and relevant in the realistic setting of actual financial markets where transaction costs and trading restrictions are significant. Indeed, in volatile financial markets like the present one, a large number of stocks, especially volatile technology stocks, become difficult and even impossible to sell short (i.e. borrow) and there are significant transaction costs associated with trading in these stocks for individual investors and even for market makers. The Black-Scholes prices for options on these stocks, i.e. the values assuming that there are no transaction costs and trading restrictions, are therefore no longer reliable. In markets such as these, it is therefore imperative to consider the problem of option pricing in the presence of market imperfections such as transaction costs and short-selling constraints to obtain reliable estimates for option prices.

The paper relates the problem of pricing options in the presence of general transaction costs and short-selling constraints to an associated pair of *stochastic impulse control* problems. The general problem of impulse control has been thoroughly investigated in the mathematical literature. (see Bensoussan [2], Bensoussan and Lions [3] and the references cited therein). Eastham and Hastings [10] and Hastings [12] provided one of the first rigorous adaptations of the theory to the problem of optimal impulse control of portfolios. Some of the other relevant papers where impulse control techniques have been applied to problems of this type are Korn [16], Bielecki and Pliska [4] and the references cited therein. This paper, for the first time, applies the theory of stochastic impulse control to the problem of option pricing in the presence of general transaction costs. After proving a set of verification theorems for the existence of optimal solutions to the impulse control problems, the paper carries out an explicit analysis of the option pricing problem when the utility function is linear and proves the existence of optimal policies and provides an explicit characterization of them. This analysis culminates in the design of a numerical procedure to calculate the price of a european option on the stock which is then compared with the Black-Scholes price. The paper then considers the situation where the investor's utility function is general and proves an approximation theorem for the existence of the value function. This is used to provide a qualitative characterization of the option price corresponding to an exponential and a linear-power utility function.

The analysis of the present paper is related to the paper by Davis, Panas and Zariphopoulou [8] from an economic standpoint in that the preference-based model used by them is employed here. Their paper, however dealt with the problem of option pricing in the presence of proportional transaction costs and related it to the methodology of *singular stochastic control* as opposed to the stochastic impulse control methodology of the present paper. From a mathematical standpoint, the paper is closely related to the papers by Eastham and Hastings [10], Hastings [12], Korn [16] and Bielecki and Pliska [4] in the impulse control methodology adopted. The problem that the present paper considers is, however, significantly different from the ones considered in the aforementioned papers.

The outline of the paper is as follows.

In Section 2, we provide a rigorous introduction to the problem of option pricing and define the price of a European option in terms of the value functions of two stochastic optimization problems. As in Davis et al. [8], we define the replicating portfolio for a European option and prove that when the space of admissible trading strategies is linear, the price of the option is the initial endowment associated with the replicating portfolio and therefore reduces to the Black-Scholes price. We also present, for the first time, a generalized definition of the option price applicable to general utility functions which reduces to the Black-Scholes price in frictionless markets and provides a suitable modification of the Black-Scholes price in the presence of market frictions.

In Section 3, we carry out a detailed investigation of the situation with general transaction costs and short-selling constraints and show that it suffices to restrict the admissible trading strategies to those involving only a finite number of transactions over the time horizon with probability one. We thereby demonstrate the correspondence between the optimization problem introduced in Section 2 and a stochastic impulse control problem. We also prove verification theorems for the existence of optimal policies under fairly general conditions on the price processes and transaction costs structure.

In Section 4, we consider a market where the drifts and volatilities of the bond and stock are constants as in the Black-Scholes market, the transaction costs are a combination of proportional and fixed costs, there are short-selling constraints on the bond and stock, and the investor's utility function is linear. We explicitly prove the intuitive result that the optimal trading policy for the investor involves at most one intermediate transaction over the time horizon. Using these results, we implement the Crank-Nicolson finite-difference scheme to numerically solve the quasi-variational inequalities associated with the impulse control problem to obtain the price of a call option on the stock and compare it with the Black-Scholes price to illustrate the effects of transaction costs and short-selling constraints.

In Section 5, we discuss the situation where the investor's utility function has a general form and prove that the value function is the pointwise limit of an increasing sequence of value functions of associated optimal stopping problems. This result is reminiscent of the traditional technique in the theory of stochastic impulse control of "iterating upon the obstacle" (see Bensoussan and Lions [3]) but those results cannot be directly applied in our setting since the value functions we encounter do not satisfy the regularity hypotheses required there.

This enables us to devise a numerical approximation procedure to solve the impulse control problem in this general setting and thereby obtain the price of the option. We also carry out a qualitative investigation of the option price in the presence of exponential and linear-power utility functions.

Section 6 concludes the paper.

2. OPTION PRICING

2.1. Option Pricing as a Utility Maximization Problem

In this section, we adapt the ideas of Davis et al. [8] to our setting and give a definition of the option price in terms of a utility maximization problem. In order to facilitate comparison with their definitions and results, we retain their notation whenever possible.

$\mathbf{P} = (P_0, P_1, \ldots, P_n)$ is a vector-valued stochastic process describing the prices of a risk-free asset or bond P_0 and n risky assets or stocks. These are defined on a probability space (Ω, F, Q) and a time interval $[0, T]$. The right continuous complete and augmented filtration F_t of the probability space is generated by the price process $\mathbf{P}(t)$. We shall define a price at time zero for a European option with exercise time T on one of the stocks, say $P_1(t)$.

Let $\Gamma(S_0)$ denote the set of *admissible trading strategies* for an investor who starts with S_0 shares of the risk-free asset and no shares in stocks. At this stage, we assume that the notion of admissibility has been defined unambiguously. We shall later give a precise definition of an admissible trading strategy in the context of the specific problem we are considering.

At any time t, the value of the investor's portfolio is given by

$$V_t = \mathbf{S}(t).\mathbf{P}(t),$$

where $\mathbf{S}(t)$ is the vector-valued "share" process. If the investor carries out a transaction at time $t \in [0, T)$, then

$$V_{t+} = \mathbf{S}(t).\mathbf{P}(t) - \Psi(\mathbf{P}(t), \mathbf{S}(t), \mathbf{S}'(t)) - \beta,$$

where $\beta > 0$ and $\mathbf{S}'(t)$ is the share process after the transaction. At this stage, we assume that Ψ is a continuous, nonnegative function of its arguments.

Thus, $\beta + \Psi(\mathbf{P}(t), \mathbf{S}(t), \mathbf{S}'(t))$ represents the total transaction cost accompanying the transaction.

In this paper, we shall make the following important assumption about the nature of the market.

Assumption: Transaction costs are incurred *before* carrying out a transaction and only accompany a *rebalancing* of the portfolio. Since the portfolio is liquidated at the terminal date T, no transaction costs are incurred then.

We assume that the European option on the stock $P_1(t)$ is given by a nonnegative continuous *payoff function* $\eta(P_1(T))$. The option writer forms a portfolio to hedge the option and liquidates it at time T. Thus, the value of the writer's portfolio after he has met his liabilities is $\mathbf{P}.\mathbf{S}(T) - \eta(P_1(T))$.

A call option on the stock $P_1(t)$ is the right to buy one share at time T at a price E. The option writer forms a portfolio to hedge the option and liquidates it at time T. If $P_1 \leq E$ the option is not exercised and the cash value of the portfolio is $\mathbf{P}.\mathbf{S}(T)$. If $P_1 > E$, the buyer pays the writer E in cash, and the writer gives one share to the buyer. The cash value of the portfolio is therefore $\mathbf{S}(t).\mathbf{P}(t) + E - P_1(T)$. Therefore, in this case,

$$\eta(P_1(T)) = I_{P_1(T)>E}(P_1(T) - E)$$

Let $U : R \to \bar{R}$ be the writer's utility function which is concave and increasing where $\bar{R} = R \cup \{-\infty, \infty\}$ is the extended real line. We assume that $U(0) \equiv 0$ and we also assume throughout the paper that

$$E[|U(-\eta(P_1(T)))|] < \infty.$$

i.e. we will only consider utility functions U and payoff functions $\eta(\cdot)$ satisfying the above inequality.

Just as in Davis et al. [8], we can define the following value functions:

$$V_w(S_0) = \sup_{\pi \in \Gamma(S_0)} E[U((\mathbf{P}.\mathbf{S}^{\pi}(T) - \eta(P_1(T)))] \tag{2.1}$$

and

$$V_1(S_0) = \sup_{\pi \in \Gamma(S_0)} E[U((\mathbf{P}.\mathbf{S}^{\pi}(T)], \tag{2.2}$$

where $\Gamma(S_0)$ is the set of all admissible trading strategies with initial endowment S_0 and π is an admissible trading strategy. $V_w(S_0)$ is the maximum utility available to the writer if he writes the option and $V_1(S_0)$ is the maximum utility if he enters the market with an initial endowment of S_0 shares of the risk-free asset and does not write the option. Let

$$\rho_w = \inf\{S_0 : V_w(S_0) \geq 0\},$$

and

$$\rho_1 = \inf\{S_0 : V_1(S_0) \geq 0\}.$$

Just as in Davis et al. [8], we can define the price of the option as

$$P_w = \rho_w - \rho_1.$$

A *replicating portfolio* for the option contract is an element $\pi_0 \in \Gamma(S_0)$ (if it exists) and an initial endowment S_0 such that $\mathbf{P}.\mathbf{S}(T) = \eta(P_1(T))$, where we do not want to have any excess cash remaining at time T.

The following proposition is exactly analogous to Theorem 1 in Davis et al. [8].

Proposition 2.1. *Suppose Γ is a linear space, i.e. $\pi_1 \in \Gamma(S_1), \pi_2 \in \Gamma(S_2) \Longrightarrow \pi_1 + \pi_2 \in \Gamma(S_1 + S_2)$ and $V_w(S)$ and $V_1(S)$ are both continuous and strictly increasing functions of S. Then $p_w = S_0$ if a replicating portfolio $\pi_0 \in \Gamma(S_0)$ exists.*

Proof. Exactly as in Davis et al. [8].

Just as in the paper by Davis et al. [8], we see that the Black-Scholes model satisfies the conditions of Proposition 2.1. Therefore, in the absence of transaction costs and constraints and for a utility function satisfying the conditions of the proposition, the price of a European option reduces to the Black-Scholes price.

2.2. Generalized Definition of the Option Price

An important assumption underlying the preceding definition of the option price is that the relevant portfolio optimization problems must have unique, well-defined solutions. This assumption is somewhat restrictive since we cannot define the option price for more general utility functions. In this subsection, we shall generalize the definition in order to be able to consider more general utility functions.

Let $U : R \to R$ be a utility function that is concave and increasing with $U(0) = 0$. The reader may recall that we have assumed that

$$E[|U(-\eta(P_1(T)))|] < \infty.$$

Let $\Gamma(\mathbf{p}, \mathbf{s}, t)$ be the set of *admissible trading strategies* available to the investor in the absence of transaction costs and constraints where we again assume that the notion of *admissibility* has been defined unambiguously. If

$$\sup_{\pi \in \Gamma(\mathbf{p},\mathbf{s},t)} E_{\mathbf{p},\mathbf{s},t} U(\mathbf{P}.\mathbf{S}(T)) \equiv u(\mathbf{p}, \mathbf{s}, t) < \infty \forall \mathbf{p}, \mathbf{s}, t, \tag{2.3}$$

and

$$\sup_{\pi \in \Gamma(\mathbf{p},\mathbf{s},t)} E_{\mathbf{p},\mathbf{s},t} U(\mathbf{P}.\mathbf{S}(T) - \eta(P_1(T))) \equiv \nu(\mathbf{p}, \mathbf{s}, t) < \infty \forall \mathbf{p}, \mathbf{s}, t, \tag{2.4}$$

then it is easy to see that the above problems have well-defined solutions for the set Γ' of trading strategies with transaction costs and constraints, so that the option price can be uniquely defined using the definitions in the preceding subsection.

In particular, we note that if the utility function $U(\cdot)$ is bounded above, (2.3) and (2.4) have unique, well-defined solutions and the option price can therefore be unambiguously defined. An example of such a utility function is the exponential function

$$U(x) = 1 - \exp(-\gamma x), \quad \text{for } 0 < \gamma < \infty.$$

For a general utility function $U(\cdot)$ with $U(x) < \infty$ for $x \in R$ define

$$V_M(x) = U(x) 1_{x<M} + U(M) 1_{x \geq M}$$

For each $M > 0$, $V_M(.)$ is clearly concave, increasing with $V_M(0) = 0$. Since $V_M(.)$ is bounded above for each $M > 0$, the problems (2.3) and (2.4) have unique well-defined solutions and therefore the option price p_M in the presence of transaction costs and constraints can be unambiguously defined. We define the option price for the utility function $U(\cdot)$ as follows :

$$p = \lim_{M \to \infty} p_M \ \text{ if it exists.}$$

In particular, we note that if the conditions of Proposition 2.1 are satisfied each p_M is the Black-Scholes price for the option so that the price p is also the Black-Scholes price. Thus, our definition of the option price for the more general utility function $U(\cdot)$ is a valid generalization of the Black-Scholes price. The motivation for the *regularization procedure* employed in the above definition is the realization that the option price essentially depends on the behavior of the utility function in a neighborhood of the origin. (see Davis et al. [8]). Further, the result of the following proposition essentially implies that if the option price is defined in this generalized sense, then the option prices in the presence of market frictions reduce to the Black-Scholes price when the frictions are removed.

The following proposition proves that the generalized definition of the option price is consistent with the original definition, i.e. if U is a utility function for which (2.3) and (2.4) have unique well-defined solutions for a given set of admissible trading strategies Γ (which may include market frictions in general) then the two definitions of the option price coincide.

Proposition 2.2. *Let U be a utility function for which* (2.3) *and* (2.4) *have unique, well-defined solutions with respect to a set of admissible trading strategies Γ. Let U_n be a sequence of utility functions that increases pointwise to U. Suppose further that the value functions V_1, V_w, V_{1n}, V_{wn} corresponding to U and U_n respectively are continuous and strictly increasing. Let $p < \infty$ and $p_n < \infty$ be the option prices associated with U and U_n respectively. Then*

$$\lim_{n\to\infty} p_n = p$$

Proof: Firstly, we note that since $U_n(x) \le U(x)$ for all x, (2.3) and (2.4) have unique well-defined solutions for each U_n and therefore the option prices p_n exist. Let

$$V_1(S_0) = \sup_{\pi\in\Gamma(S_0)} E[U(\mathbf{P}.\mathbf{S}^\pi(T))],$$
$$V_{1n}(S_0) = \sup_{\pi\in\Gamma(S_0)} E[U_n(\mathbf{P}.\mathbf{S}^\pi(T))],$$
$$V_w(S_0) = \sup_{\pi\in\Gamma(S_0)} E[U(\mathbf{P}.\mathbf{S}^\pi(T) - \eta(P_1(T))],$$
$$V_{wn}(S_0) = \sup_{\pi\in\Gamma(S_0)} E[U_n(\mathbf{P}.\mathbf{S}^\pi(T) - \eta(P_1(T))],$$

By hypothesis, all the functions above are continuous and strictly increasing.

We shall first prove that

$$\lim_{n\to\infty} V_{1n}(S_0) = V_1(S_0)$$
$$\lim_{n\to\infty} V_{wn}(S_0) = V_w(S_0).$$

Since $U_n \uparrow U$, we see that

$$\begin{aligned}\lim_{n\to\infty} V_{1n}(S_0) &= \sup_n \sup_{\pi\in\Gamma(S_0)} E[U_n(\mathbf{P}.\mathbf{S}^\pi(T))]\\ &= \sup_{\pi\in\Gamma(S_0)} \sup_n E[U_n(\mathbf{P}.\mathbf{S}^\pi(T))]\\ &= \sup_{\pi\in\Gamma(S_0)} \lim_{n\to\infty} E[U_n(\mathbf{P}.\mathbf{S}^\pi(T))]\\ &= \sup_{\pi\in\Gamma(S_0)} E[U(\mathbf{P}.\mathbf{S}^\pi(T))]\end{aligned}$$

by the monotone convergence theorem. The proof for the other case follows in a similar fashion if we use the dominated convergence theorem and the assumption that $E[|U(-\eta(P_1(T)))|] < \infty$ and $E[|U_n(-\eta(P_1(T)))|] < \infty$. Therefore $V_{1n} \uparrow V_1$ pointwise and $V_{wn} \uparrow V_w$ pointwise. Let

$$\rho_1 = \inf\{S_0 : V_1(S_0) \geq 0\},$$
$$\rho_{1n} = \inf\{S_0 : V_{1n}(S_0) \geq 0\}.$$

It suffices to prove that $\lim_{n\to\infty} \rho_{1n} = \rho_1$. The proof for the other case is similar.

Firstly, we observe that since $V_{1n} \uparrow V_1$, $\rho_{1n} \geq \rho_1$ for all n and S_{1n} decreases with n. Let $\lim_{n\to\infty} \rho_{1n} = \rho_1^* \geq \rho_1$. We need to show that $\rho_1^* = \rho_1$.

Note: ρ_1^* exist since $\rho_1 < \infty$ by assumption.

Consider any $\rho > S_1$. Since V_1 is strictly increasing by hypothesis, $V_1(\rho) > 0$. Since $V_{1n} \uparrow V_1$, there exists $N < \infty$ such that $V_{1n}(\rho) > 0$ for $n \geq N$. Since V_{1n} are continuous and strictly increasing, it follows that $S > \rho_{1n}$ for $n \geq N$. Therefore $S > \rho_1^*$. Since S is arbitrary, it follows that $\rho_1^* = \rho_1$. This completes the proof.

Comments: The reader will correctly observe that there are several different generalized definitions of the option price possible depending on the choice of the sequence of "approximating" bounded utility functions corresponding to a given utility function U. However, we have seen that in frictionless markets all these generalized definitions reduce to the Black-Scholes price. The result of the above proposition implies that if the utility function U satisfies (2.3) and (2.4) for a given set of admissible trading strategies Γ (which may include market frictions in general), the generalized definition of the option price coincides with the original one for any suitable "approximating" sequence of utility functions that are bounded above. The purpose of our generalized definition is to provide a suitable *regularization* of the original definition for a utility function U for which solutions to the optimization problems do not exist, and which reduces to the Black-Scholes price when market frictions are eliminated. For a given set of admissible trading strategies Γ, we will only consider utility functions for which (2.3) and (2.4) have solutions, so that the option price is well-defined in the sense of Davis et al. [8].

3. THE MODEL

In this section, we shall introduce a model to price a European option in the presence of general transaction costs and short-selling constraints. For the sake of simplicity of exposition, we assume a financial market with one bond and one stock.

3.1. Price Processes for the Bond and Stock

If $\mathbf{P} \equiv (P_0, P_1)$ is the vector-valued price process for the bond and stock, then we assume that

$$dP_0(r) = \mu_0(P_0(r), r)dr,$$

with $P_0(t) = p_0 > 0$ for $t \in [0, T]$ and $\mu_0(\cdot) > 0$.

$$dP_1(r) = \mu_1(P_1(r), r)dr + \sigma(P_1(r), r)dW(r).$$

The initial condition is given by $P_1(t) = p_1 > 0$ for $t \in [0, T]$. The functions μ_0, μ_1, σ are all assumed to be uniformly bounded, deterministic functions of their arguments. Under the above conditions, the stock price is a diffusion process.

3.2. The Admissible Trading Strategies

The admissible trading strategies for the option writer can be described as follows. At any time $t \in [0, T)$, the investor holds a portfolio of shares of each asset represented by the vector $\mathbf{S}(t) \in R^2$. He intervenes at various instants to rebalance his portfolio and at each time of intervention, he pays a transaction cost so that the value of his portfolio satisfies:

$$\mathbf{P}.\mathbf{S}'(t) = \mathbf{P}.\mathbf{S}(t) - \Psi(\mathbf{P}(t), \mathbf{S}(t), \mathbf{S}'(t)) - \beta,$$

where $t \in [0, T)$ is a time at which he intervenes or trades and $\mathbf{S}$ and $\mathbf{S}'$ are the initial and final share vectors respectively.

The function $\Psi(\mathbf{P}(t), \mathbf{S}(t), \mathbf{S}'(t))$ represents the transaction costs accompanying the transaction. At this stage, this function is assumed to be a continuous, nonnegative function of its arguments. In the next section, we shall choose the function Ψ to be the sum of the proportional and fixed transaction costs described in the introduction. At the final time T he dissolves his portfolio and consumes it with no transaction costs.

We assume that there are short-selling constraints on the bond and the stock. In this paper, we assume that $\mathbf{S}(t) \in R^2_+$. The results of the paper can be extended to the situation when the number of shares that can be sold short is nonzero but bounded.

Following the notation of Eastham and Hastings [10], let $\mathbf{B} \equiv R^2_+$. The *feasible* set $K(\mathbf{P}, \mathbf{S})$ corresponding to the price vector $\mathbf{P}$ and share vector $\mathbf{S}$ is given by

$$K(\mathbf{P}, \mathbf{S}) \equiv \{\mathbf{S}' \in \mathbf{B} : \mathbf{P}.\mathbf{S}' = \mathbf{P}.\mathbf{S} - \Psi(\mathbf{P}, \mathbf{S}, \mathbf{S}') - \beta\}.$$

Let $\mathbf{K} = \{(\mathbf{P}, \mathbf{S}) : K(\mathbf{P}, \mathbf{S}) \neq \emptyset\}$. From the above definitions, it is clear that $\mathbf{K}$ is a closed set.

Note: It is important to observe that the admissible trading strategies include those strategies involving continuous trading in $[0, T]$. Therefore, in the absence of transaction costs and short-selling constraints, a *replicating portfolio* for the option exists by the martingale representation theorem. The result of Proposition 2.1 holds for a utility function satisfying the conditions of the proposition and also for a more general utility function by using the generalized definition of the option price given in section 2.2. In a subsequent subsection, we shall prove that in the presence of fixed transaction costs and short-selling constraints, the admissible trading strategies cannot involve an infinite number of transactions in $[0, T]$.

3.3. The Impulse Control Problem

From the model and the definitions introduced in the previous section, we shall see that the investor's optimization problem is an *impulse control* problem (Bensoussan and Lions [3], Eastham and Hastings [10]).

An *impulse control* $\nu \equiv \{(\theta_k, \mathbf{S}_k)\}$ is a sequence such that (θ_k) is an increasing sequence of stopping times and $(\mathbf{S}_k)$ is a sequence of R^2-valued random variables such that $\mathbf{S}_k \in F_{\theta_k}$ where F_t is the complete, right continuous filtration on the underlying probability space. If $\mathbf{X}$ is the joint price-share process $(\mathbf{P}, \mathbf{S})$, the impulse control ν is called *admissible* if

$$\theta_k \text{ is a stopping time of } F_t,$$
$$\theta_k \le \theta_{k+1} \text{ for any } k,$$
$$P[\lim_{k\to\infty} \theta_k \le T] = 0,$$
$$\mathbf{S}_k \in K(\mathbf{P}(\theta_k), \mathbf{S}_{k-1}),$$
$$\mathbf{S}_k \in F_{\theta_k}.$$

If U is the investor's utility function, then we can define the value of the admissible impulse control ν by

$$J(\mathbf{p}, \mathbf{s}, t, \nu) = E_{\mathbf{p},\mathbf{s},t}[U(C_T)], \tag{3.1}$$

where $C_T \in R$ is the consumption at the terminal date T, $\mathbf{P}(t) = \mathbf{p}$ and $\mathbf{S}(t) = \mathbf{s}$. The above expression represents the expected utility of consumption at the terminal date when the investor starts out at time t with the price process $\mathbf{P}(t) = \mathbf{p}$ and the share process $\mathbf{S}(t) = \mathbf{s}$ and follows a trading strategy defined by the impulse control policy ν.

In our situation,

$$C_T = \mathbf{P}(T).\mathbf{S}(T),$$

in the calculation of $V_1(\cdot)$ and

$$C_T = \mathbf{P}(T).\mathbf{S}(T) - \eta(P_1(T)),$$

in the calculation of $V_w(\cdot)$.

Note: In the problem considered in this paper, there is consumption only at the terminal time T in contrast with the portfolio selection problem studied by Eastham and Hastings [10] and Hastings [12].

Further, the "consumption" at the terminal date could be negative in the calculation of V_w due to the nonnegative option payoff $\eta(P_1(T))$.

The impulse control problem reduces to finding an admissible control ν^* such that

$$J(\mathbf{p}, \mathbf{s}, t, \nu^*) \ge J(\mathbf{p}, \mathbf{s}, t, \nu) \text{ for each } \mathbf{p} \in R_+^2, \mathbf{s} \in \mathbf{B}, \nu \text{ admissible}.$$

The following proposition proves that if the utility function U is such that (2.3) and (2.4) have unique, well-defined solutions with respect to the set Γ of admissible trading strategies with short-selling prohibitions but no transaction costs, then the admissible trading strategies in the presence of strictly positive transaction costs can only involve a finite number of transactions almost surely over the time horizon.

Proposition 3.1. *If the investor's utility function* $U(\cdot)$ *is such that* (2.3) *and* (2.4) *have unique*, *well-defined solutions with respect to the set of admissible trading strategies with short-selling prohibitions but no transaction costs then the admissible trading strategies in the presence of strictly positive transaction costs in addition to short-selling prohibitions involve only a finite number of transactions almost surely in* $[0, T]$.

Proof: It suffices for us to consider the problem of maximizing

$$J(\mathbf{p}, \mathbf{s}, t, \nu)E_{p,s,t}[U(\mathbf{P}(T).\mathbf{S}(T))],$$

over all admissible trading strategies ν. The problem for the other value function can be handled in a similar fashion using the assumption that $E[|U(-\eta(P_1(T)))|] < \infty$.

In the absence of transaction costs, the optimization problem above has a finite solution $u(\mathbf{p}, \mathbf{s}, t)$ by assumption. Since $J(\mathbf{p}, \mathbf{s}, t, \nu)$ is the value function in the presence of transaction costs,

$$J(\mathbf{p}, \mathbf{s}, t, \nu) \leq u(\mathbf{p}, \mathbf{s}, t) < \infty \forall \mathbf{p}, \mathbf{s}, t, \nu.$$

Suppose ν^* is an admissible trading strategy in the presence of short-selling prohibitions and strictly positive transaction costs. We need to prove that ν^* cannot involve an infinite number of transactions in $[0, T]$ with positive probability.

For each $\omega \in \Omega$, $\nu^*(\omega)$ is a set of trading times. Let

$$D = \{\omega \in \Omega, \mathit{card}(\nu^*(\omega)) = \infty\}.$$

We need to show that $P(D) = 0$. Suppose, to the contrary, that $P(D) > 0$. Let

$$F = \{\omega \in \Omega : \mathbf{P}(T, \omega)\bar{\mathbf{S}}(T, \omega) < \infty\}.$$

for the policy ν^* and in the absence of transaction costs where $\bar{\mathbf{S}}(.)$ is the share process for the policy ν^* assuming no transaction costs. Clearly, $P(F) = 1$, since the price process $\mathbf{P}$ is almost surely finite and the investor cannot hold an infinite number of shares and the optimization problem in the absence of transaction costs is assumed to have a solution.

Therefore $P(D \cap F) = P(D) > 0$. Since each transaction is accompanied by the strictly positive transaction cost, at any transaction instant t_n

$$\mathbf{P}(t_n).\mathbf{S}(t_n) \leq \mathbf{P}(t_n).\bar{\mathbf{S}}(t_n) - \beta$$

Therefore, it is easy to see that on the set $D \cap F$,

$$\mathbf{P}(T, \omega).\mathbf{S}(T, \omega) = -\infty.$$

where $\mathbf{S}(\cdot)$. is the terminal share process in the presence of transaction costs.
But this contradicts the fact that $\mathbf{P}(T, \omega).\mathbf{S}(T, \omega) \geq 0$.

This contradiction implies that $P(D \cap F) = P(D) = 0$ and the assertion of the proposition is proved.

Note: The linear utility function $U(\dot{x}) \equiv x$ satisfies the conditions of the above proposition when the market coefficients μ_0, μ_1, σ are uniformly bounded above and below away from zero. In the presence of short-selling constraints, (2.3) and (2.4) have unique, well-defined solutions for this utility function.

For a general utility function, we can use the generalized definition of the option price presented in section 2.2 to restrict consideration to utility functions satisfying the conditions of the above proposition. It therefore suffices for us to restrict the space of *admissible trading strategies* to those involving only a finite number of transactions almost surely in $[0, T]$

We therefore see that the admissible impulse controls correspond to the admissible trading strategies available to the investor since an admissible trading strategy is a monotonic increasing sequence of *trading instants* $\{\theta_n\}$ which are stopping times of the filtration, and an associated sequence of *share vectors* $\{\mathbf{S}_n\}$ which represent the holdings of the investor at the respective instants θ_n.

By definition, the option price is expressed in terms of the functions V_w and V_1 introduced in equations (2.1) and (2.2). Both V_w and V_1 are solutions of optimization problems over the set of admissible trading strategies which, in our case, is the set of admissible impulse controls. From equations (2.1) and (2.2), it is clear that the function to be optimized in either case has exactly the same form as in equation (3.1) with the consumption C_T being the argument of the utility function $U(\cdot)$ in either case. Therefore, the impulse control problem is precisely the optimization problem we need to solve in order to determine the price of the option. In the next subsection, we shall introduce the quasi-variational inequalities that describe the solution of the impulse control problem and characterize the value function.

Note: It is important to realize that the result of Proposition 3.1 relies heavily on the fact that $\beta > 0$, i.e. transaction costs are strictly positive.

3.4. The Optimal Impulse Control Policy

We shall now introduce the quasi-variational inequalities satisfied by a sufficiently regular solution of the impulse control problem. Let $u = u(\mathbf{p}, \mathbf{s}, r)$ be a continuous real-valued function on $R_+^2 \times \mathbf{B} \times R_+$ with continuous first partial derivatives in r and continuous second partial derivatives in $\mathbf{p}$. We define the operator

$$\mathbf{L}u(\mathbf{p}, \mathbf{s}, r) \equiv u_r(\mathbf{p}, \mathbf{s}, r) + \nabla u(\mathbf{p}, \mathbf{s}, r).\mu(\mathbf{p}, r) + (1/2)\sigma^2 u_{11}(\mathbf{p}, \mathbf{s}, r),$$

The subscripts denote partial derivatives with respect to p_1 and ∇ denotes the gradient operator with respect to $\mathbf{p}$.

We also define the operator

$$\mathbf{M}u(\mathbf{p}, \mathbf{s}, r) = \sup_{s' \in K(\mathbf{p}, s)} u(\mathbf{p}, \mathbf{s}', r).$$

If $K(\mathbf{p}, \mathbf{s}) = \emptyset$, we define $\mathbf{M}u(\mathbf{p}, \mathbf{s}, r) = -\infty$.

We note that the operator $\mathbf{M}$ is well defined since for each $(\mathbf{p}, \mathbf{s}) \in \mathbf{K}$, $K(\mathbf{p}, \mathbf{s})$ is compact and non-empty. Further, using arguments similar to those in Eastham and Hastings [10], it can be shown that K is upper semicontinuous as a multifunction. Since u is continuous and $\mathbf{B}$ is locally compact, a measurable selection theorem due to Schael [21] tells us that there exists a Borel measurable function $\varphi : \mathbf{K} \times [0, T] \to \mathbf{B}$ such that

$$\varphi(\mathbf{p}, \mathbf{s}, r) \in K(\mathbf{p}, \mathbf{s}),$$

and

$$\mathbf{M}u(\mathbf{p}, \mathbf{s}, r) = u(\mathbf{p}, \varphi(\mathbf{p}, \mathbf{s}, r), r),$$

for all $(\mathbf{p}, \mathbf{s}, r) \in R_+^2 \times \mathbf{B} \times [0, T)$.

Let C be the space of continuous, real-valued functions on $R_+^2 \times \mathbf{B} \times R_+$ with continuous first partial derivatives in r, continuous second partial derivatives in $\mathbf{p}$ and satisfying **Dynkin's formula**, i.e.

$$E_{\mathbf{p},\mathbf{s},t} u(\mathbf{p}(\tau), \mathbf{s}(\tau), \tau) = u(\mathbf{p}, \mathbf{s}, t) + E_{\mathbf{p},\mathbf{s},t} \int_t^\tau \mathbf{L}u(\mathbf{p}, \mathbf{s}, r) dr.$$

Suppose there exists $u \in C$ such that

$$u \geq \mathbf{M}u, \mathbf{L}u \leq 0,$$
$$(u - \mathbf{M}u)\mathbf{L}u = 0,$$
$$u(\mathbf{p}, \mathbf{s}, T) = U(C_T(\mathbf{p}, \mathbf{s})),$$

where

$$C_T(\mathbf{p}, \mathbf{s}) = \mathbf{p}.\mathbf{s},$$

in the calculation of V_1 and

$$C_T(\mathbf{p}, \mathbf{s}) = \mathbf{p}.\mathbf{s} - \eta(P_1(T)),$$

in the calculation of V_w.

We can define the *continuation* set $\mathbf{C} = \{u > \mathbf{M}u\}$ and the action set $\mathbf{A} = \mathbf{C}^c$.

We can construct the control policy generated by u as follows.

Using the notation of Eastham and Hastings [10] the initial data is $(\mathbf{p}, \mathbf{s}, t)$ we set $\theta_0^* = t$ and $\mathbf{S}_0^* = \mathbf{s}$. Define

$$\theta_k^* = \inf\{r \geq \theta_{k-1}^* : (\mathbf{P}(r), \mathbf{S}_{k-1}^*, r) \in \mathbf{A}\},$$

where $\theta_k^* = \infty$ if the process does not enter **A** before T.

$$\mathbf{S}_k^* = \varphi(\mathbf{P}(\theta_k^*), \mathbf{S}_{k-1}^*, r).$$

By the upper semicontinuity of $\mathbf{M}u$ and the continuity of u, it is easy to see that **A** is a closed set. Therefore, θ_k^* is a stopping time for each k. By the continuity of the price process, $\mathbf{P}(\theta_k^*), \mathbf{S}_{k-1}^*, \theta_k^*) \in \mathbf{A}$ and $\mathbf{S}_k^*$ is well-defined.

The following propositions provide an important characterization of a sufficiently regular solution of the quasi-variational inequalities.

Proposition 3.2. *u is the minimum element of the set of functions $\nu \in C$ satisfying*

$$\nu \geq \mathbf{M}u, \mathbf{L}\nu \leq 0,$$
$$\nu(\mathbf{p}, \mathbf{s}, T) = U(C_T(\mathbf{p}, \mathbf{s}, T))$$

where U is a continuous function.

Proof: Since u is continuous and $\mathbf{M}u$ is upper semicontinuous, the *continuation set* $\mathbf{C} = \{u > \mathbf{M}u\}$ is an open set. Let $\nu \in C$ be as in the statement of the proposition. Clearly, $\nu \geq \mathbf{M}u$ implies that $\nu \geq u$ on $\mathbf{C}^c$. It remains to show that $\nu \geq u$ on the set **C**.

Suppose we assume the contrary, i.e. there exists $(\mathbf{p}, \mathbf{s}, r) \in \mathbf{B} \times [0, T)$ such that $u(\mathbf{p}, \mathbf{s}, r) > \nu(\mathbf{p}, \mathbf{s}, r)$. Since $\mathbf{L}u = 0$ on $u > \mathbf{M}u$ and $\mathbf{L}\nu \leq 0$, it follows that $\mathbf{L}(u - \nu) \geq 0$ on $\{u > \mathbf{M}u\}$.

Define $\tau = \inf\{t > r : u(\mathbf{p}(t), \mathbf{s}(t), t) = \nu(\mathbf{p}(t), \mathbf{s}(t), t)\}$ with $\mathbf{p}(r) = \mathbf{p}$ and $\mathbf{s}(r) = \mathbf{s}(t) = \mathbf{s}$. Since $u(\mathbf{p}, \mathbf{s}, T) = \nu(\mathbf{p}, \mathbf{s}, T)$, clearly $\tau \leq T$. Further τ is a stopping time since u and ν are continuous functions and therefore $\{u > \nu\}$ is an open set. Applying Dynkin's formula to $u - \nu$, we see that

$$E_r(u - \nu)(\tau) = (u - \nu)(r) + E_r \int_r^\tau \mathbf{L}(u - \nu)(s)ds$$

where the other arguments of $u - \nu$ have been suppressed. By the definition of τ, and the fact that $(u - \nu)(r) > 0$, it follows that we must have $\mathbf{L}(u - \nu)(s) < 0$ for some $s \in [r, \tau)$. Therefore, $\mathbf{L}\nu(s) > 0$ which contradicts the assumptions on ν Hence, $\nu \geq u$ on **C**. This completes the proof.

Note: In the context of the problem we are considering, we shall use the following proposition.

Proposition 3.3. *Suppose $u \in C$ is a sufficiently regular solution of the quasi-variational inequalities giving rise to an admissible control policy. Let ν be a continuous function satisfying where $\tau_1, \tau_2 \leq T$ are stopping times. Then $\nu \geq u$. If u' is the value function of any admissible control policy, then $\nu \geq u'$.*

Proof: The proof is along the same lines as the proof of the first part of Theorem 3.2 in Eastham and Hastings [10]

Proposition 3.4. *Suppose u is a sufficiently regular solution of the quasi-variational inequalities giving rise to an admissible control* $\nu^* \equiv \{(\theta_k^*, \mathbf{S}_k^*)\}$. *Also suppose that u satisfies Dynkin's formula, i.e.*

$$E[u(\mathbf{p}(\theta_k'), \mathbf{S}_{k-1}, \theta_k') - u(\mathbf{P}(\theta_{k-1}', \mathbf{S}_{k-1}, \theta_{k-1}')] = E\int_{\theta_{k-1}'}^{\theta_k'} \mathbf{L}u(\mathbf{P}(r), \mathbf{S}_{k-1}, r)dr,$$

where $\nu \equiv \{(\theta_k, \mathbf{S}_k)\}$ is any admissible impulse control and $\theta_k' = \theta_k \wedge T$. Then ν^* is an optimal impulse control.

Proof: The proof of this proposition follows exactly along the lines of the proof of Theorem 3.2 in Eastham and Hastings [10]

3.5. Relationship Between the Option Price and the Hedging Price

Let Γ be the space of all trading strategies assuming the presence of *short-selling* prohibitions but no transaction costs. The results obtained thus far tell us that the price of a European option is well-defined(in the sense of Davis et al. [8]) for any utility function U for which (2.3) and (2.4) have unique, well-defined solutions relative to Γ. In particular, we note that the linear utility function $U(x) \equiv x$ satisfies this condition.

If Γ' is the space of admissible trading strategies with transaction costs and short-selling prohibitions, then clearly a utility function satisfying (2.3) and (2.4) with respect to Γ satisfies (2.3) and (2.4) with respect to Γ'. We can therefore define(in the sense of Davis et al. [8]) the price p of a European contingent claim represented by the strictly nonnegative time T payoff $\eta(T)$ in the presence of transaction costs and short-selling constraints for an investor with the utility function U. Further, the result of Proposition 2.2 implies that this definition coincides with the *generalized* definition we have introduced in the previous section.

The *hedging price* p_h of a European contingent claim with nonnegative time T payoff is defined as follows:

$$p_h \equiv \inf\{S_0 : \exists \nu \in \Gamma'(S_0); W^\nu(T) \geq \eta(T) a.s.\}$$

where $\Gamma'(S_0)$ is the space of all trading strategies in the presence of short-selling constraints and transaction costs starting out with S_0 shares of the risk-free asset.

Define

$$\bar{U}(x) = U(x), x \geq 0,$$
$$\bar{U}(x) = -\infty, x < 0.$$

Since $\bar{U}(x) \leq U(x)$, (2.3) and (2.4) have unique, well-defined solutions relative to Γ'. Therefore, the option price $p_{\bar{U}}$ corresponding to the utility function $\bar{U}$ and in the presence of transaction costs and short-selling prohibitions is well-defined. (It could be equal to infinity).

Proposition 3.5. *Let $p_{\bar{U}}$ be the price of the European option corresponding to the utility function $\bar{U}$ and in the presence of transaction costs and short-selling prohibitions. Then $p_{\bar{U}} = p_h$ where p_h is the hedging price of the option.*

Proof: In the notation of section 2 it is easy to see that

$$S_1 = \inf\{S_0 : V_1(S_0) \geq 0\} = 0.$$
$$S_w = \inf\{S_0 : V_w(S_0) \geq 0\},$$

where

$$V_w(S_0) = \sup_{\pi \in \Gamma'(S_0)} E[\bar{U}(\mathbf{P}.\mathbf{S}(T) - \eta(P_1(T))]$$

From the definition of $\bar{U}$, it is easy to see that $V_w(S_0) = -\infty$ unless $\mathbf{P}.\mathbf{S}(T) \geq \eta(P_1(T))$ a.s. in which case $V_w(S_0) \geq 0$. It follows easily that

$$p_{\bar{U}} = S_w - S_1 = S_w = p_h.$$

This completes the proof.

Proposition 3.6. *Let p_U be the price of the European option corresponding to U. Then $p_U \leq p_{\bar{U}} = p_h$. Thus, the option price corresponding to the utility function U in the presence of short-selling prohibitions and transaction costs provides a nontrivial lower bound on the hedging price of the option.*

Proof: By definition,

$$p_U = \inf\{S_0 : V_w(S_0) \geq 0\},$$

where

$$V_w(S_0) = \sup_{\pi \in \Gamma'(S_0)} E[U(\mathbf{P}.\mathbf{S}(T) - \eta(P_1(T)))]$$
$$\geq \sup_{\pi \in \Gamma'(S_0)} E[\bar{U}(\mathbf{P}.\mathbf{S}(T) - \eta(P_1(T)))]$$

Since $p_{\bar{U}} = \inf\{S_0 : \bar{V}_w(S_0) \geq 0\}$ where $\bar{V}_w(S_0)$ is the supremum on the right hand side above, we see that $p_U \leq p_{\bar{U}}$. This completes the proof.

Note: In particular, we note that the linear utility function $U(x) \equiv x$ satisfies (2.3) and (2.4) with respect to the space Γ of admissible trading strategies in the presence of short-selling prohibitions but no transaction costs and therefore with respect to the space Γ' of trading strategies with transaction costs in addition to short-selling prohibitions. The results obtained thus far imply that the option price for the linear utility function in the presence of transaction costs and short-selling prohibitions can be defined in the sense of Davis et al. [8] and this definition coincides with the *generalized definition* introduced in section 2.

The result of Proposition 2.2 tells us that if U_n is any increasing sequence of utility functions converging pointwise to U satisfying the conditions of the proposition then

$$\lim_{n\to\infty} p_{U_n} = p_U$$

In general, the option price for the linear utility function provides a nontrivial lower bound on the option prices (in the presence of transaction costs and short-selling constraints) corresponding to all utility functions bounded above by the linear utility function. In the next section, we shall carry out an explicit calculation of p_U by solving the associated stochastic impulse control problem.

Note: The optimization problem we consider differs from the ones studied by Hastings [12] and Eastham and Hastings [10] in the fact that there is consumption only at the terminal date and also in the general transaction cost structure investigated.

4. EUROPEAN OPTION PRICE FOR THE LINEAR UTILITY FUNCTION

In this section, we shall use the results of the previous section to calculate the price of a European option for an investor with a linear utility function in the situation where the parameters of the bond and stock price processes are constants and the transaction costs are a combination of *proportional* and fixed costs and there are *short-selling* prohibitions on the bond and the stock. We shall then compare the price of the option with the Black-Scholes price and thus demonstrate the effect of transaction costs.

4.1. Price Processes for the Bond and Stock

The price processes for the bond and stock are given by

$$dP_0(t) = \mu_0 P_0(t)dt,$$
$$dP_1(t) = \mu_1 P_1(t)dt + \sigma P_1(t)dW(t),$$

We assume throughout that $\mu_1 > \mu_0 \geq 0$.

The partial differential operator associated with the optimal impulse control policy is given by

$$\mathbf{L}u = u_t + \mu_0 p_0 u_{p0} + \mu_1 p_1 u_{p1} + (1/2)\sigma^2 p_1^2 u_{p_1 p_1}.$$

In this section, we shall assume that the investor has the utility function $U(x) \equiv x$. We assume that the transaction costs have the following form

$$\Psi(\mathbf{p}, \mathbf{s}, \mathbf{s}') + \beta = (1-\alpha)|\mathbf{p}.\mathbf{s}| + \lambda|p_1(s_1 - s_1')| + \lambda_0|p_0(s_0 - s_0')| + \beta$$

where $0 < \alpha \leq 1$, $0 \leq \lambda, \lambda_0 < 1$, $\beta > 0$.

In the notation of the previous section, we have

$$\mathbf{B} \equiv R_+^2$$

and

$$K(\mathbf{p}, \mathbf{s}) = \{\mathbf{s}' \in \mathbf{B} : \mathbf{p}.\mathbf{s}.' = \mathbf{p}.\mathbf{s} - \Psi(\mathbf{p}, \mathbf{s}, \mathbf{s}')\},$$
$$K = \{(\mathbf{p}, \mathbf{s}) : K(\mathbf{p}, \mathbf{s}) \neq \emptyset\}.$$

We recall that

$$\mathbf{M}u(\mathbf{p}, \mathbf{s}, t) = \sup_{\mathbf{s}' \in K(\mathbf{p},\mathbf{s})} u(\mathbf{p}, \mathbf{s}', t) \text{ if } K(\mathbf{p}, \mathbf{s}) \neq \emptyset$$
$$= -\infty \quad \text{if } K(\mathbf{p}, \mathbf{s}) = \emptyset.$$

4.2. The Impulse Control Problems

By the results of previous sections, the value functions $V_w(\cdot)$ and $V_1(\cdot)$ are the solutions of the following problems:

$$V_1(S_0) = \sup_{\pi \in \Gamma(S_0)} E[\mathbf{P}.\mathbf{S}(T)],$$
$$V_w(S_0) = \sup_{\pi \in \Gamma(S_0)} E[\mathbf{P}.\mathbf{S}(T) - \eta(P_1(T))].$$

If we set

$$\delta = E[\eta(P_1(T))],$$

then, since the value of δ is independent of the trading strategy of the investor, we see that

$$V_w(S_0) = V_1(S_0) - \delta.$$

Therefore,

$$S_1 = \inf\{S_0 : V_1(S_0) \geq 0\},$$
$$S_w = \inf\{S_0 : V_1(S_0) \geq \eta\}.$$

Thus, we only need to calculate $V_1(\cdot)$.

4.3. Calculation of $\mathbf{V_1(S_0)}$

In this subsection, we shall calculate the value function $V_1(S_0)$. We shall prove that the optimal impulse control policy involves *at most* one intermediate transaction in $[0.T)$.

In the process, we shall provide an explicit characterization of the value function $V_1(S_0)$.

By the results of section 3, the quasi-variational inequalities associated with the impulse control problem are given by

$$u \geq \mathbf{M}u, \mathbf{L}u \geq 0,$$
$$(u - \mathbf{M}u)\mathbf{L}u = 0,$$
$$u(\mathbf{p}, \mathbf{s}, T) = \mathbf{p}.\mathbf{s}$$

We shall attempt to solve the above *implicit obstacle* problem by the usual method of "iterating upon the obstacle". (see e.g. Eastham and Hastings [10]). We choose u_0 as the solution of the following:

$$\mathbf{L}u_0 = 0,$$
$$u_0(\mathbf{p}, \mathbf{s}, T) = \mathbf{p}.\mathbf{s}.$$

Define $u_i, i = 1, 2.3, \ldots$ by

$$\mathbf{L}u_i \leq 0, u_i \geq u_{i-1},$$
$$(\mathbf{L}u_i)(u_i - \mathbf{M}u_{i-1}) = 0,$$
$$u_i(\mathbf{p}, \mathbf{s}, T) = \mathbf{p}.\mathbf{s}.$$

It is easy to see that

$$u_0(\mathbf{p}, \mathbf{s}, t) = p_0 s_0 \exp(\mu_0(T - t)) + p_1 s_1 \exp(\mu_1(T - t)).$$

We shall now prove a lemma that will be used in the proof of the main proposition.

Lemma 4.1.

$$\mathbf{M}u_0(\mathbf{p}, \mathbf{s}, t) = \left(\frac{\alpha(\mathbf{p}.\mathbf{s}) - \beta - \lambda p_1 s_1 - \lambda_0 p_0 s_0}{1 - \lambda}\right) \exp(\mu_1(T - t))$$

if $\alpha(\mathbf{p}.\mathbf{s}) - \beta < p_1 s_1 + \lambda_0 p_0 s_0$,

$$= \left(\frac{\alpha(\mathbf{p}.\mathbf{s}) - \beta - p_1 s_1 - \lambda_0 p_0 s_0}{1 - \lambda_0}\right) \exp(\mu_0(T - t)) + p_1 s_1 \exp(\mu_1(T - t))$$

if $\alpha(\mathbf{p}.\mathbf{s}) - \beta \geq p_1 s_1 + \lambda_0 p_0 s_0, t \geq T_0$

$$= \left(\frac{\alpha(\mathbf{p}.\mathbf{s}) - \beta + \lambda p_1 s_1 - \lambda_0 p_0 s_0}{1 - \lambda}\right) \exp(\mu_1(T - t))$$

if $\alpha(\mathbf{p}.\mathbf{s}) - \beta \geq p_1 s_1 + \lambda_0 p_0 s_0, t < T_0$

where $T_0 = T - (\mu_1 - \mu_0)^{-1} \log\left(\frac{1 + \lambda}{1 - \lambda_0}\right)$

Proof:

$$u_0 = p_0 s_0 \exp(\mu_0(T - t)) + p_1 s_1 \exp(\mu_1(T - t))$$
$$\mathbf{M}u_0(\mathbf{p}, \mathbf{s}, t) = \sup_{\mathbf{s}' \in K(\mathbf{p}, \mathbf{s})} u_0(\mathbf{p}, \mathbf{s}', t).$$

If we define

$$\bar{K}(\mathbf{p}, \mathbf{s}) \equiv \left\{\mathbf{s}' \in R^2_+ : \mathbf{p}.\mathbf{s}' \leq \alpha\mathbf{p}.\mathbf{s} - \beta - \lambda_0 p_0|s_0 - s'_0| - \lambda p_1|s_1 - s'_1|\right\},$$

it is easy to see that since $u_0(\cdot)$ is a linear function of $\mathbf{s}$,

$$\mathbf{M}u_0(\mathbf{p}, \mathbf{s}, t) = \sup_{\mathbf{s}' \in \bar{K}(\mathbf{p},\mathbf{s})} u_0(\mathbf{p}, \mathbf{s}', t).$$

Further, since $\bar{K}(\mathbf{p}, \mathbf{s})$ is a convex set, the supremum above is attained at its extreme points. The three feasible extreme points are easily seen to satisfy

$$\mathbf{p}.\mathbf{s}' = \left(0, \frac{\alpha\mathbf{p}.\mathbf{s} - \beta - \lambda_0 p_0 s_0 + \lambda p_1 s_1}{1+\lambda}\right) \text{ if } \alpha\mathbf{p}.\mathbf{s} - \beta \geq p_1 s_1 + \lambda_0 p_0 s_0,$$

$$\mathbf{p}.\mathbf{s}' = \left(0, \frac{\alpha\mathbf{p}.\mathbf{s} - \beta - \lambda_0 p_0 s_0 - \lambda p_1 s_1}{1-\lambda}\right), \text{ if } \alpha\mathbf{p}.\mathbf{s} - < p_1 s_1 + \lambda_0 p_0 s_0,$$

$$\mathbf{p}.\mathbf{s}' = \left(\frac{\alpha\mathbf{p}.\mathbf{s} - \beta - \lambda_0 p_0 s_0 - p_1 s_1}{1+\lambda_0}, p_1 s_1\right).$$

Comparing the values of $u_0(\cdot)$ at these extreme points, we easily obtain the result of the lemma.

We shall now prove the main proposition of the section.

Proposition 4.1. *The optimal trading policy involves at most one transaction in* $[0, T)$*, i.e.* $u = u_1$.

Proof: We shall prove the assertion by first constructing a supersolution $\bar{u}$ to the quasi-variational inequalities associated with the impulse control problem. Define

$$\begin{aligned} \bar{u} &= u_0 \text{ if } t \geq T_0 \\ &= \left(\frac{\mathbf{p}.\mathbf{s} + \lambda p_1 s_1 - \lambda_0 p_0 s_0}{1+\lambda}\right) \exp(\mu_1(T-t)) \text{ if } t < T_0. \end{aligned}$$

Using arguments similar to those used in the proof of the previous lemma, we can show that $\mathbf{M}\bar{u} = \mathbf{M}u_0$.

We shall now prove that $\bar{u} \geq \mathbf{M}\bar{u}$.

For $t \geq T_0$, $\bar{u} = u_0 = p_0 s_0 \exp(\mu_0(T-t)) + p_1 s_1 \exp(\mu_1(T-t))$

For $t \geq T_0$ and $\alpha\mathbf{p}.\mathbf{s} - \beta \geq p_1 s_1 + \lambda_0 p_0 s_0$

$$\begin{aligned} \mathbf{M}\bar{u}(\mathbf{p}, \mathbf{s}, t) = &\left(\frac{\alpha\mathbf{p}.\mathbf{s} - \beta - p_1 s_1 - \lambda_0 p_0 s_0}{1-\lambda_0}\right) \exp(\mu_0(T-t)) \\ &+ p_1 s_1 \exp(\mu_1(T-t))] \end{aligned}$$

Clearly, $\bar{u} \geq \mathbf{M}\bar{u}$ in this case.

For $t \geq T_0$ and $\alpha\mathbf{p}.\mathbf{s} - \beta < p_1 s_1 + \lambda_0 p_0 s_0$

$$\mathbf{M}\bar{u}(\mathbf{p}, \mathbf{s}, t) = \left(\frac{\alpha\mathbf{p}.\mathbf{s} - \beta - \lambda p_1 s_1 - \lambda_0 p_0 s_0}{1-\lambda}\right) \exp(\mu_1(T-t)).$$

$\bar{u} \geq \mathbf{M}\bar{u}$ in this case since

$$(\alpha \mathbf{p}.\mathbf{s} - \beta) \exp(\mu_1(T-t)) < (p_1 s_1 + \lambda_0 p_0 s_0) \exp(\mu_1(T-t))$$
$$\leq (p_1 s_1 + \lambda_0 p_0 s_0) \exp(\mu_1(T-t)) + (1-\lambda) \exp(\mu_0(T-t)).$$

For $t < T_0$ and $\alpha \mathbf{p}.\mathbf{s} - \beta \geq p_1 s_1 + \lambda_0 p_0 s_0$

$$\mathbf{M}\bar{u}(\mathbf{p}, \mathbf{s}, t) = \left(\frac{\alpha \mathbf{p}.\mathbf{s} - \beta + \lambda p_1 s_1 - \lambda_0 p_0 s_0}{1+\lambda}\right) \exp(\mu_1(T-t)) \text{ and}$$
$$\bar{u} = \left(\frac{\mathbf{p}.\mathbf{s} + \lambda p_1 s_1 - \lambda_0 p_0 s_0}{1+\lambda}\right) \exp(\mu_1(T-t)).$$

Clearly $\bar{u} \geq \mathbf{M}\bar{u}$ in this case.
For $t < T_0$ and $\alpha \mathbf{p}.\mathbf{s} - \beta < p_1 s_1$

$$\mathbf{M}\bar{u}(\mathbf{p}, \mathbf{s}, t) = \left(\frac{\alpha \mathbf{p}.\mathbf{s} - \beta - \lambda p_1 s_1 - \lambda_0 p_0 s_0}{1+\lambda}\right) \exp(\mu_1(T-t)).$$

$\bar{u} \geq \mathbf{M}\bar{u}$ in this case since

$$2\lambda(p_1 s_1 + \lambda_0 p_0 s_0) \geq 2\lambda(\alpha \mathbf{p}.\mathbf{s} - \beta).$$

Next we see that $\mathbf{L}\bar{u} \leq 0$ except at $t = T_0$ where it is undefined. However an application of Ito's lemma and the continuity of $\bar{u}$ at T_0 imply that

$$E[\bar{u}(\mathbf{P}(\tau_2), \mathbf{s}, \tau_2) - \bar{u}(\mathbf{P}(\tau_1), \mathbf{s}, \tau_1) \leq 0,$$

where $\tau_1 \leq \tau_2 \leq T$ are stopping times. We can now use the result of Proposition 3.3 to conclude that $\bar{u} \geq u_1$ where u_1 is the value function corresponding to control policies allowing at most one intervention in $[0, T)$.

Since the operator $\mathbf{M}$ is monotone, $\mathbf{M}\bar{u} \geq \mathbf{M}u_1 \geq \mathbf{M}u_0$. It follows that $\mathbf{M}u_1 = \mathbf{M}u_0$. From the definition of u_1, it therefore satisfies

$$u_1 \geq \mathbf{M}u_1, \mathbf{L}u_1 \leq 0,$$
$$(u_1 - \mathbf{M}u_1), \mathbf{L}u_1 = 0,$$
$$u_1(\mathbf{p}, \mathbf{s}, T) = \mathbf{p}.\mathbf{s}.$$

Since the control policy generated by u_1 is clearly admissible, u_1 is the value function of the impulse control problem. This completes the proof of the proposition.

Remark: It is important to realize that the result of the proposition is true for arbitrary $\alpha \in (0, 1]$, $\lambda_0, \lambda \in (0, 1)$. In particular, it is true for $\alpha = 1$ and $\lambda_0 > \lambda_1$ which is the situation where proportional transaction costs in the bond exceed those in the stock. Further, the optimal policy in the frictionless case of transferring all wealth to the stock immediately is *not* the optimal policy in the problem with transaction costs, in general.

4.4. Numerical Calculation of the Option Price

The results of the previous subsection reduce the stochastic impulse control problem for calculating the option price to an optimal stopping problem since the optimal trading policy requires at most one intervention in $[0, T)$. This problem can be solved by solving the following system of *variational inequalities*:

$$u_1 \geq \mathbf{M}u_0, \mathbf{L}u_1 \leq 0,$$
$$(u_1 - \mathbf{M}u_0), \mathbf{L}u_1 = 0,$$
$$u_1(\mathbf{p}, \mathbf{s}, T) = \mathbf{p}.\mathbf{s}.$$

where $u_0(\mathbf{p}, \mathbf{s}, t) = p_0 s_0 \exp(\mu_0(T - t)) + p_1 s_1 \exp(\mu_1(T - t))$ and $\mathbf{M}u_0$ is as obtained in Lemma 4.1. The price of the European option can then be obtained in terms of u_1 as follows:

$$\rho_1 = \inf\{S_0 : u_1(\mathbf{p}, S_0, S_1, t) \geq 0\} \text{ with } S_1 = 0$$

and

$$\rho_w = \inf\{S_0 : u_1(\mathbf{p}, S_0, S_1, t) \geq \delta\} \text{ with } S_1 = 0$$

where

$$\delta = E_{\mathbf{p},t}[\eta(P_1(T)].$$

In the presence of short-selling constraints, it is easy to see that $\rho_1 = 0$. Therefore the price of the European option is ρ_w.

The system of variational inequalities may be numerically solved by the well-known Markov Chain Approximation technique and the numerical solutions may be shown to converge to the value function of the original optimal stopping problem as the approximation parameter goes to zero. (see Kushner and Dupuis [17]).

We implemented the Markov Chain Approximation technique to calculate the price of an ordinary European call option on the stock for the linear utility function in the presence of general transaction costs and short-selling constraints. The results are shown in the figure titled ‘Option Price for Linear Utility Function’. The bold curve represents the option price for the linear utility function in the presence of short-selling prohibitions and transaction costs and the light curve represents the Black-Scholes price. We notice that the option price is always greater than the Black-Scholes price as expected.

5. OPTION PRICING FOR GENERAL UTILITY FUNCTIONS

In this section, we shall investigate the option pricing problem when the investor’s utility function is more general.

Let Γ be the set of all trading strategies for the investor over the time horizon in the presence of short-selling prohibitions and in the absence of any transaction costs. Throughout this section, we shall assume that the investor’s utility function

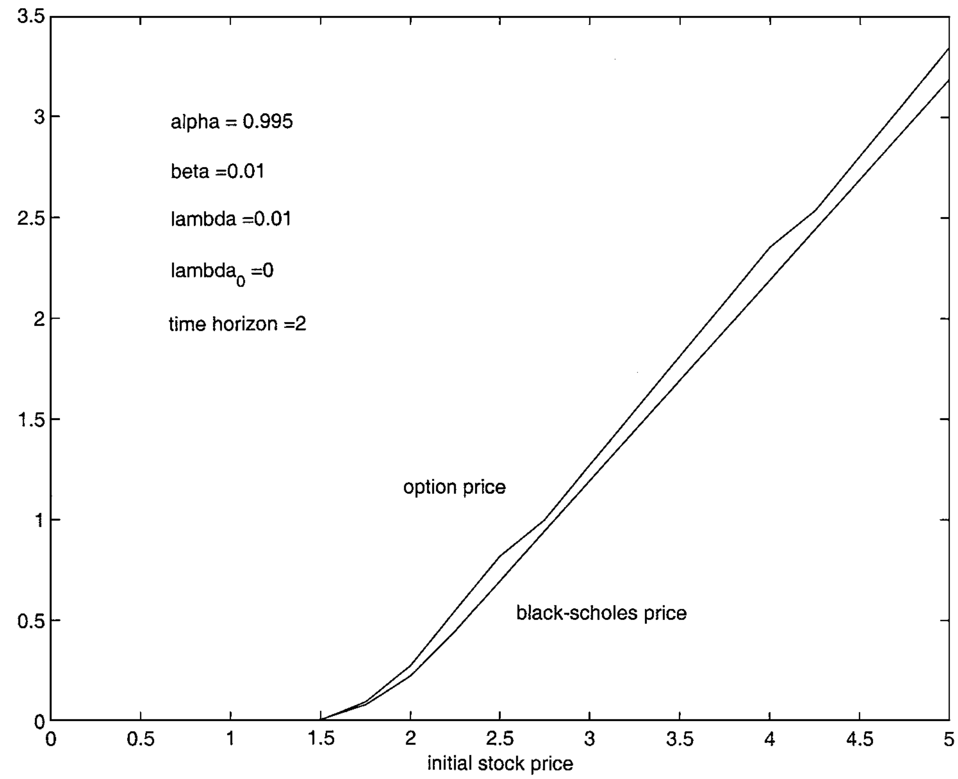

Figure 1. Option price for linear utility function.

U is a continuous, concave, monotonic increasing function such that (2.3) and (2.4) have solutions with respect to Γ. Therefore,

$$\sup_{\pi\in\Gamma(\mathbf{p},\mathbf{s},t)} E[U(\mathbf{P}.\mathbf{S}^{\pi}(T))] = \bar{u}(\mathbf{p},\mathbf{s},t) < \infty \tag{5.1}$$

$$\sup_{\pi\in\Gamma(\mathbf{p},\mathbf{s},t)} E[U(\mathbf{P}.\mathbf{S}^{\pi}(T) - \eta(P_1(T))] = \bar{v}(\mathbf{p},\mathbf{s},t) < \infty \tag{5.2}$$

Let Γ' be the set of trading strategies for the investor in the presence of short-selling prohibitions and strictly positive transaction costs. By the result of Proposition 3.1 it is enough to restrict Γ' to the set of trading strategies involving only a finite number of trades almost surely over the time horizon.

Define

$$u(\mathbf{p},\mathbf{s},t) = \sup_{\pi\in\Gamma'(\mathbf{p},\mathbf{s},t)} E[U(\mathbf{P}(T).\mathbf{S}^{\pi}(T))],$$

$$u_i(\mathbf{p},\mathbf{s},t) = \sup_{\pi\in\Gamma_i'(\mathbf{p},\mathbf{s},t)} E[U(\mathbf{P}(T).\mathbf{S}^{\pi}(T))],$$

$$w(\mathbf{p},\mathbf{s},t) = \sup_{\pi\in\Gamma'(\mathbf{p},\mathbf{s},t)} E[U(\mathbf{P}(T).\mathbf{S}^{\pi}(T) - \eta(P_1(T)))],$$

$$w_i(\mathbf{p}, \mathbf{s}, t) = \sup_{\pi \in \Gamma_i'(\mathbf{p},\mathbf{s},t)} E[U(\mathbf{P}(T).\mathbf{S}^{\pi}(T) - \eta(P_1(T)))],$$

where $\Gamma_i' \subset \Gamma'$ consists of those strategies involving at most i trades over the time horizon. Clearly, $u_i \leq u_{i+1} \leq u$, $w_i \leq w_{i+1} \leq w$. Further, the value functions u, u_i, w and w_i exist since (5.1) and (5.2) hold.

Recall our standing assumption that

$$E[|U(-\eta(P_1(T))|] = \delta' < \infty. \tag{5.3}$$

Proposition 5.1. *If the functions* $u,\ u_i,\ w,\ w_i$ *are as defined above then*

$\lim_{i\to\infty} u_i(\mathbf{p}, \mathbf{s}, t) = u(\mathbf{p}, \mathbf{s}, t)$. *for every* $\mathbf{p}, \mathbf{s}, t$

$\lim_{i\to\infty} w_i(\mathbf{p}, \mathbf{s}, t) = w(\mathbf{p}, \mathbf{s}, t)$. *for every* $\mathbf{p}, \mathbf{s}, t$

Proof:

Part 1: $\lim_{i\to\infty} u_i = u$

Since $u_i(.) \leq u_{i+1}(.) \leq u$ for each i, $\lim_{i\to\infty} u_i(\mathbf{p}, \mathbf{s}, t) = \nu(\mathbf{p}, \mathbf{s}, t)$ exist for each **p**,**s**,**t** and $\nu(.) \leq u(.)$. We need to show that $\nu(.) \equiv u(.)$.

Suppose $\exists(\mathbf{p}, \mathbf{s}, t)|\nu(\mathbf{p}, \mathbf{s}, t) < u(\mathbf{p}, \mathbf{s}, t)$.

Since $u(\mathbf{p}, \mathbf{s}, t) = \sup_{\pi\in\Gamma'(\mathbf{p},\mathbf{s},t)} E[U(\mathbf{P}(T).\mathbf{S}^{\pi}(T)]$, given any $\varepsilon > 0$, $\exists \pi^* \in \Gamma'$ such that

$$E[U(\mathbf{P}(T).\mathbf{S}^{\pi^*}(T))] \geq u(\mathbf{p}, \mathbf{s}, t) - \varepsilon.$$

Since ε is arbitrary, we can choose it so that $\nu(\mathbf{p}, \mathbf{s}, t) < u(\mathbf{p}, \mathbf{s}, t) - \varepsilon$. Therefore

$$\nu(\mathbf{p}, \mathbf{s}, t) < E[U(\mathbf{P}(T).\mathbf{S}^{\pi^*}(T))].$$

The number of interventions implied by π^* is almost surely finite in $[0, T]$. Let w_j be the policy that agrees with π^* up through the *jth* intervention, then intervenes no more. Let $\mathbf{S}_j(T)$ be the terminal share process for the policy w_j. Let

$$R \equiv U(\mathbf{P}(T).\mathbf{S}^{\pi^*}(T)) \quad \text{and } R_j = U(\mathbf{P}(T).\mathbf{S}_j(T)).$$

Then, since $w_j \in \Gamma_j'$, $E[R_j] \leq u_j$. Therefore,

$$E[U(\mathbf{P}(T).\mathbf{S}^{\pi^*}(T))] - u_j(\mathbf{p}, \mathbf{s}, t) \leq E[R - R_j]$$

If (t_k) are the trading instants corresponding to policy π^*,

$$\lim_{k\to\infty} t_k \geq T \text{ a.s.}$$

since π^* involves only a finite number of transactions almost surely. Therefore

$$\lim_{j\to\infty} R - R_j = 0 \text{ a.s}$$

Further, $|R - R_j| \leq |R| + |R_j| = R + R_j$ since $R \geq 0$ and $R_j \geq 0$. If $\bar{u}(\mathbf{p}, \mathbf{s}, t)$ is as defined in (5.1), we see that

$$E[R + R_j] \leq 2\bar{u}(\mathbf{p}, \mathbf{s}, t) < \infty.$$

By the *dominated convergence theorem*, $\lim_{j\to\infty} E[R - R_j] = 0$. Therefore

$$\lim_{j\to\infty} E[R] - u_j(\mathbf{p}, \mathbf{s}, t) = E[R] - \nu(\mathbf{p}, \mathbf{s}, t) \leq 0.$$

Therefore,

$$\nu(\mathbf{p}, \mathbf{s}, t) \geq E[R] = E_{\mathbf{p},\mathbf{s},t} U(\mathbf{P}(\mathbf{T}).\mathbf{S}^{\pi^*}(T))$$

which is a contradiction by the choice of π^*.

Therefore $u(\mathbf{p}, \mathbf{s}, t) \equiv \nu(\mathbf{p}, \mathbf{s}, t)$.

Therefore $\lim_{i\to\infty} u_i \equiv u$

Part 2: $\lim_{i\to\infty} w_i = w$

The proof of this assertion follows along the lines of the proof of **Part 1** if we make use of assumption (5.3).

This completes the proof of the proposition.

Remark: The result of the above proposition does not follow trivially from the standard approximation procedure in the theory of impulse control since that procedure assumes that the *limit* function is sufficiently regular which is not true, in general, in this case.

The above proposition tells us that we can approximate the value functions u and w required to solve the option pricing problem in terms of the sequences of value functions u_i, w_i to simpler problems. In fact, it is well known that the functions u_i, w_i are successively obtained by the following general procedure: u_0 is the solution to

$$\mathbf{L}u_0 = 0,$$
$$u_0(\mathbf{p}, \mathbf{s}, T) = U((\mathbf{p}.\mathbf{s})).$$

For $i \geq 1$, u_i, is the solution to

$$u_i \geq \mathbf{M}u_{i-1}, \mathbf{L}u_i \leq 0,$$
$$(u_i - \mathbf{M}u_{i-1})\mathbf{L}u_i = 0,$$
$$u_i(\mathbf{p}, \mathbf{s}, T) = \mathbf{p}.\mathbf{s}$$

Similar assertions hold for the w_i's with $\mathbf{p}.\mathbf{s} - \eta(P_1(T))$ replacing $\mathbf{p}.\mathbf{s}$.

Each successive stage in the above procedure is an optimal stopping problem. The systems of variational inequalities above can be solved numerically by standard Markov Chain Approximation techniques as described in [17]. Further, these numerical approximations can be shown to converge to the required value functions as the approximation parameter goes to zero.

Unfortunately, the numerical procedure above is computationally intensive since the optimal trading policy for a general utility function might require a large number of transactions over the time horizon so that we would need to iterate the procedure above several times to obtain a reasonable approximation to the value function.

We have implemented this procedure to calculate the option prices for the class of exponential utility functions.

Option Pricing for Exponential Utility Functions

The general exponential utility function is of the form

$$U_\gamma(x) \equiv 1 - \exp(-\gamma x),$$

where $0 < \gamma < \infty$. Since these functions are bounded above for all γ, it is easy to see that (5.1) and (5.2) have solutions for *any* set of admissible trading strategies. Hence, the result of Proposition 5.1 holds and the option price can be obtained in terms of the solutions to the associated optimal stopping problems.

As before, in the following, we shall consider the set of admissible trading strategies Γ' with short-selling prohibitions and transaction costs. In this case, it is easy to see that the option price is given by

$$p_\gamma = \inf\left\{S_0 : \sup_{\pi \in \Gamma'} E[U_\gamma(\mathbf{P}(T).\mathbf{S}^\pi(T) - \eta(P_1(T))] \geq 0\right\}$$

where the trading strategies start out with S_0 shares of the risk-free asset.

Let p_h be the *hedging price* of the option in the market as defined in section 3.

Let p be the option price for the linear utility function obtained in section 4.

Proposition 5.2. *$p_\gamma \leq p_h$ for $0 < \gamma < \infty$ and $p \leq p_r$ for $1 \leq \gamma < \infty$.*

Proof: The first assertion follows from the result of Proposition 3.6.

It remains to prove that $p \leq p_\gamma$ for $1 \leq \gamma < \infty$

This assertion follows immediately from the fact that $U_\gamma(x) \leq x$ for $1 < \gamma < \infty$ and therefore

$$\sup_{\pi \in \Gamma'} E\Big[U_\gamma(\mathbf{P}(T).\mathbf{S}^\pi(T) - \eta(P_1(T))) \leq \sup_{\pi \in \Gamma'} E[(\mathbf{P}(T).\mathbf{S}^\pi(T) - \eta(P_1(T))\Big].$$

This completes the proof.

We have implemented the numerical approximation procedure to calculate the option price for the class of exponential utility functions. The approximations are crude due to the lack of computational power at our disposal. The results are indicated in the figure titled ‘Option Price for Exponential Utility Function’. We have chosen the utility function $U(x) \equiv 1 - \exp(-0.5x)$. The bold curve represents the option price in the presence of transaction costs and short-selling prohibitions and the light curve represents the Black-Scholes price. The option price is always greater than the Black-Scholes price as expected.

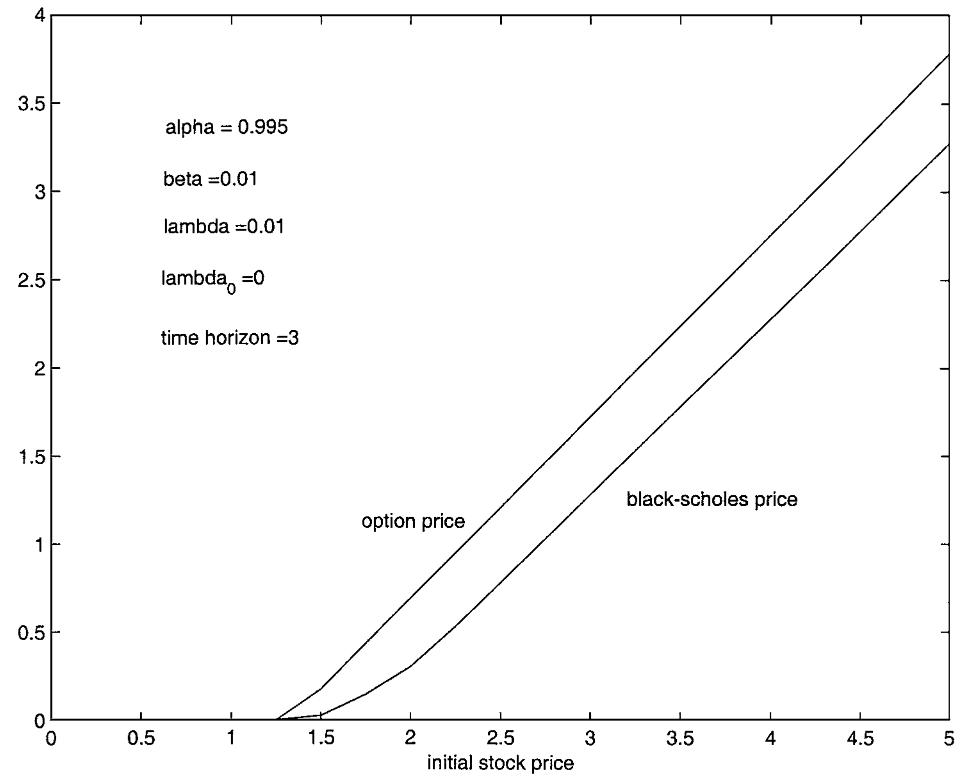

Figure 2. Option price for exponential utility function.

Option Pricing for Linear-Power Utility Functions

We now consider the class of utility functions $U_r(.)$ defined as follows:

$$U_r(x) = x, x \leq 1,$$
$$U_r(x) = x^r, x > 1$$

where $0 < r \leq 1$.

We see that $U_r(.)$ is continuous for each r. Further, $U_r(\cdot)$ is nondecreasing in r and

$$\lim_{r \to 1} U_r(x) = U_1(x) = x.$$

By the results of the previous section, we know that the option price p for the linear utility function in the presence of transaction costs and short-selling prohibitions exists and is finite. If p_r is the option price for the utility function $U_r(\cdot)$ in the presence of general transaction costs and short-selling prohibitions, the result of Proposition 2.2 tells us that p_r decreases with r and

$$\lim_{r \to 1} p_r = p$$

Therefore, the option price for the linear utility function obtained in the previous section is a good approximation to the option price for a utility function $U_r(.)$ for values of r sufficiently close to 1.

It is easy to see that the hypotheses of Proposition 5.1 are satisfied for each utility function $U_r(.)$ and for an option satisfying

$$E[|U_r(-\eta(P_1(T)))|] = E[\eta(P_1(T))] < \infty.$$

Therefore, the result of the proposition holds and the option price can be obtained in terms of the sequence of associated optimal stopping problems.

6. CONCLUSIONS AND EXTENSIONS

In this paper, we have investigated the problem of European Option Pricing in the presence of short-selling prohibitions and general transaction costs. We introduced, for the first time, a generalized definition of the option price (extending the definition of Davis et al. [8]) which reduces to the Black-Scholes price in the absence of market frictions and provides a suitable relaxation of the Black-Scholes price in the presence of market frictions. We then carried out a detailed investigation of the option pricing problem for an investor with a linear utility function in the presence of short-selling prohibitions and general transaction costs and explicitly characterized the optimal policies thereby reducing the problem to an *optimal stopping* problem. We then investigated the option pricing problem for an investor with a general utility function and proved that the price can be expressed in terms of the solutions to a sequence of optimal stopping problems. This inspired a numerical approximation procedure suitable for calculating the option price which we implemented to calculate the option price for an investor with an exponential utility function. Finally, we carried out qualitative investigations of the option prices for investors with exponential and linear-power utility functions.

The results of the paper can be easily extended to the situation where the numbers of shares of the bond and stock that can be sold short are *nonzero* and *bounded*. In the notation of the paper, we can therefore define the share-holdings constraint set $\mathbf{B}$ as

$$\mathbf{B} \equiv [-B_0, \infty) \times [-B_1, \infty) \text{ with } 0 < B_0, B_1 < \infty$$

Most of the major results of the paper can be easily shown to hold in this setting. For an investor with a linear utility function, a solution to the optimization problems exists and therefore arguments similar to those presented in Section 4 can be used to prove that the optimal policies involve at most one transaction over the time horizon thus facilitating an elementary calculation of the option price. The approximation theorem proved in Section 5 is independent of the choice of the constraint set and is therefore also true in this setting.

An extension which would require different techniques is the situation where the market coefficients are random and progressively measurable with respect to the filtration. Dynamic programming techniques can no longer be employed and one would have to resort to general martingale techniques. This is the subject of ongoing research.

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