Cat Bond Pricing Using Probability Transforms [1]

by Shaun S. Wang

1. Introduction

Catastrophe (cat) bonds are insurance-linked securities that translate referenced cat risks (natural catastrophes or man-made cats) into credit risks. In the event of a cat loss, investors, as holders of cat bond, may lose part or all of future coupon payments and principal of the bond.

Cat losses covered under the cat bond are often characterised by a loss exceedance curve, $S(x)=\Pr\{X>x\}$, that is, the probability that the cat loss X will exceed amount \$$x$. The loss exceedance curve is related to the cumulative distribution function (CDF) by $F(x)= 1-S(x)$.

For a cat bond, investors are generally provided with a loss-exceedance curve $S(x)$ that is obtained either:

- by running company exposure data through commercially available cat modelling software, or
- by designing payout functions along some parametric indicators (e.g. the Richter Scale of an earthquake at a specified location, an aggregate industry loss index, etc).

Embedded in a loss exceedance curve $S(x)$ include information of

- the expected frequency of default, and
- the recovery rate, given default.

As compensation for credit risks, just like corporate bonds, cat bonds normally offer investors yields that are higher than the risk-free interest rate (e.g. the LIBOR). The excess yields spread over the risk-free rate should more than compensate for the expected default rate, since it should also contain risk load for uncertainties associated with the default risk.

For investors, it is desirable to compare the relative attractiveness of the yields spreads between cat bonds and corporate bonds. In order to compare risk-adjusted performance of various asset classes, we would need a common yardstick that is applicable to all types of risks. For mutual funds, a popular measure of risk-adjusted performance is the Sharpe ratio, namely the excess return per unit of volatility. The Sharpe ratio works well for assets whose returns follow normal distributions. However, for a single cat bond issue, we cannot readily apply the traditional Sharpe ratio concept since the asset return is skewed and with jumps: most of the probability mass is centered at zero loss, while there is a small probability of potentially large negative returns.

In this paper, I will use probability transforms to extend the Sharpe ratio concept to credit risks with skewed return distributions, so that we can evaluate the risk-adjusted performance of the cat bond asset class. At the later part of this paper I will provide some tests of the proposed models using empirical cat bonds and corporate bonds data.

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2. Probability Transforms

The default risks of cat bonds are directly translatable to catastrophe insurance contracts. The excess yields spread over the risk-free rate for a cat bond can be translated into risk premium dollars for an insurance contract providing the same cat protection; and vise versa. For cat bond issuers, it is desirable to compare the cost of issuing a cat bond to that of purchasing equivalent reinsurance protection. Cat bond pricing presents an opportunity for reconciliation between financial and insurance pricing approaches.

In reinsurance pricing, a large risk is often subdivided into several layers. A layer in reinsurance is comparable to a call-spread in option trading, or a tranche in a cat bond series.

For a random underlying loss variable X, let $X_{(a,\ a+h]}$ denote a layer with limit h and attachment point a. The loss to the layer $X_{(a,\ a+h]}$ is related to the ground-up loss X by the following relationship:

$$X_{(a,a+h]} = \begin{cases} 0, & if \quad X < a; \\ X - a, & if \quad a \le X < a+h; \\ h, & if \quad a+h \le X. \end{cases}$$

It can be verified that the expected loss to the layer $X_{(a,\ a+h]}$ equals the area under the loss exceedance curve over the interval $(a, a+h]$:

$$E\left[X_{(a,a+h]}\right] = \int_a^{a+h} S(x)dx .$$

When the layer limit h is sufficiently small, the expected loss to the layer is

$$E\left[X_{(a,a+h]}\right] \approx S(a)\cdot h .$$

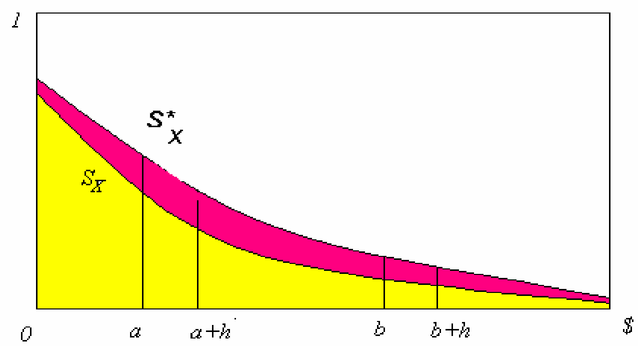
Suppose that we have an observed price, $E^*[X_{(a,\ a+h]}]$, for a small layer $(a, a+h]$. Note that the layer price $E^*[X_{(a,\ a+h]}]$ often contains a risk load in addition to the expected loss $E[X_{(a,\ a+h]}]$. Form the layer price $E^*[X_{(a,\ a+h]}]$ we can infer a price-implied loss exceedance probability:

$$S^*(a) \approx E^*\left[X_{(a,a+h]}\right]\cdot \frac{1}{h}.$$

We can expect that $S^*(a) \ge S(a)$. Indeed, observed market prices by layer imply a *direct* transform of the loss exceedance curve from $S(x)$ to $S^*(x)$.

It was with this insight that Venter (1991) argued that no-arbitrage pricing always implies a transformed distribution (see Figure 1 below).

Figure 1. Observed market prices by layer imply a direct transform of the loss exceedance curve S(x) to a "price curve" S(x).*



Inspired by Venter's observation, Wang (1996) studied a class of probability transform: $S^*(x) = g[S(x)]$, where $g[0,1] \rightarrow [0,1]$ is increasing with $g(0)=0$ and $g(1)=1$. A particularly simple probability transform is the proportional hazard transform: $S^*(x) = S(x)^{1-\lambda}$, with $0 \le \lambda < 1$. It has desirable actuarial properties but it is not directly related to financial pricing theories such as CAPM and Black-Scholes.

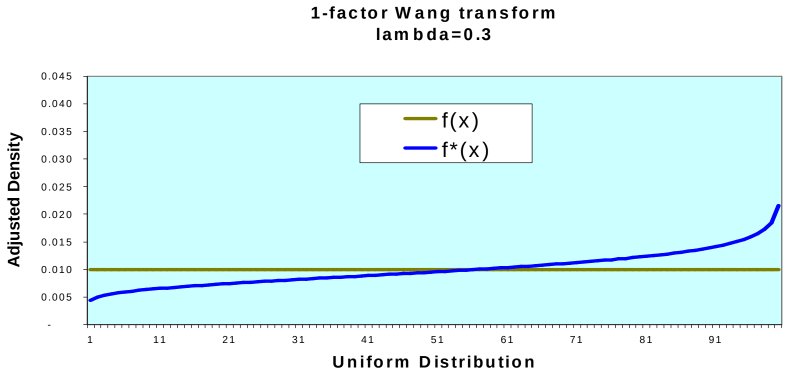
Motivated by directly extending the Sharpe ratio concept to risks with skewed distribution, the author (Wang, 2000) proposed the following Wang transform:

$$S^*(x) = \Phi\left(\Phi^{-1}(S(x)) + \lambda\right), \qquad (1)$$

Here Φ represents the standard normal cumulative distribution, although there is no restriction on the distributional form of $S(x)$. As we shall see later on, the parameter λ is a direct extension of the Sharpe Ratio.

For a given loss variable X with objective loss exceedance curve $S(x)$, the Wang transform (1) produces a "risk-adjusted" loss exceedance curve $S^*(x)$. The mean value under $S^*(x)$, denoted by $E^*[X]$, will define a risk-adjusted "fair value" of X at time T, which can be further discounted to time zero, using the risk-free interest rate.

Figure 2: The Wang transform (λ=0.3) of a Uniform[0,100] distribution (displayed in terms of probability density)



As shown in Figure 2, the Wang transform inflates probability density for adverse outcomes, while deflating probability density for favorable outcomes; as a result it incorporates a form of risk adjustment or risk loading.

Under the Wang transform, if S has a normal(μ,σ^2) distribution, S^* is also a normal distribution with $\mu^* = \mu+\lambda\sigma$ and $\sigma^* = \sigma$; If S has a lognormal(μ,σ^2) distribution such that ln(X) ~ normal(μ,σ^2), S^* is another lognormal distribution with $\mu^* = \mu+\lambda\sigma$ and $\sigma^* = \sigma$. Thus, for normally distributed risks, the parameter λ in (1) is exactly the Sharpe Ratio.
A liability with loss variable X can be viewed as a negative asset with gain variable $Y = -X$, and vice versa. Mathematically, if a liability has a market price of risk λ, when treated as a negative asset, the market price of risk will be $-\lambda$. That is, the market price of risk will have the same value but opposite signs, depending upon whether a risk vehicle is treated as an asset or liability. For an asset with gain variable X, the Wang transform (1) has an equivalent representation:

$$F^*(x) = \Phi\left[\Phi^{-1}(F(x)) + \lambda\right]$$

where $F(x) = 1-S(x)$ is the cumulative distribution function (cdf) of X. With a change in the sign of λ, we obtain the same price for both sides of a risk transaction.

3. Adjustment for Parameter Uncertainty

So far we have assumed that probability distributions for risks under consideration are known without ambiguity. In realty, we always have to estimate probability distributions based on limited available data. As a result, parameter uncertainty is always present. Even with the best data and technologies available today, there are parameter uncertainties in the modelling of catastrophe losses (see Major, 1999).

Consider the classic sampling theory in statistics. Assume that we have m independent observations from a given population with a normal(μ,σ^2) distribution. Note that μ and σ are not directly observable, we can at best estimate μ and σ by the sample mean $\tilde{\mu}$ and sample standard deviation $\tilde{\sigma}$. As a result, when we make probability assessments regarding a future outcome, we effectively need to use a student-t distribution with $k = m-2$ degrees of freedom.

Following the statistical sampling theory that uses a student-t distribution in place of a normal distribution, we suggest the following technique of adjusting for parameter uncertainty for any empirically estimated probability distribution $F(x)$:

$$F^*(x) = Q\left(\Phi^{-1}(F(x))\right),$$

or equivalently in terms of $S(x)= 1-F(x)$:

$$S^*(x) = Q\left(\Phi^{-1}(S(x))\right) \qquad (2)$$

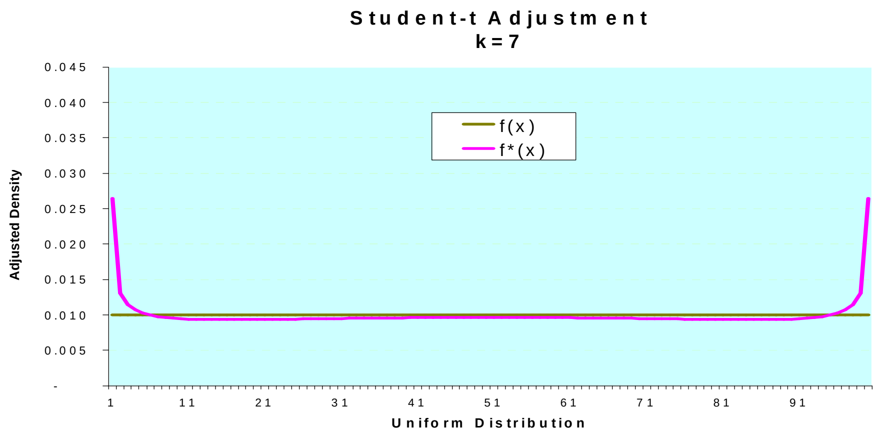
where Q has a student-t distribution with degree-of-freedom k and it a probability density

$$f(t;k) = \frac{1}{\sqrt{2\pi}} \cdot c_k \cdot \left[1 + \frac{t^2}{k}\right]^{-(0.5k+1)}, \qquad -\infty<t<\infty$$

where

$$c_k = \sqrt{\frac{2}{k}} \cdot \frac{\Gamma((k+1)/2)}{\Gamma(k/2)}.$$

Figure 3: The Student-t adjustment of a Uniform[0,100] distribution (displayed in terms of probability density)



As shown in Figure 3, the Student-t adjustment mainly inflates probability densities at both extreme tails of the Uniform[0, 100] distribution, whiling keeping the density at the central region relatively unchanged.

Let $S(x)$ be an empirically estimated probability distribution, before adjustment for parameter uncertainty. The combination of parameter uncertainty adjustment in (2) and pure risk adjustment using the Wang Transform in (1) yields the following two-factor model:

$$S^*(y) = Q\left(\Phi^{-1}(S(y)) + \lambda\right) \qquad (3)$$

where Q has a student-t distribution with degree-of-freedom k.

4. Risk Premium for Higher Moments

In classic CAPM where asset returns are assumed to follow multivariate normal distributions, the "market price of risk," or the Sharpe ratio, $\lambda=(E[R]-r)/\sigma[R]$, represents the excess return per unit of volatility.

The classic CAPM has gone through important enhancements in modern finance and insurance research. In addition to risk premium associated with volatility, there are strong evidences of risk premium for higher moments and parameter uncertainty. Kozik and Larson (2001) offer insightful discussions on the risk premium for higher moments, demonstrating that 3-moment CAPM significantly improves the fit of empirical data.

Obviously, risk premium for higher moments has direct implications in pricing property catastrophe insurance, high excess-of-loss insurance layers, credit default risk, and way-out-of-the-money options.

We can view the Wang transform (1) as an analog to the two-moment CAPM, which does not produce sufficient risk adjustment at the extreme tails of probability distributions.

The Student-t adjustment (2) captures two opposing forces that often distort investors' rational behavior, namely *greed* and *fear*. Although investors may fear unexpected large losses, they desire unexpected large gains. As a result the tail probabilities are often inflated at both tails; and the magnitude of distortion normally increases at the extreme tails. This distributional adjustment at both tails increases the kurtosis of the underlying distribution. The mean value of the transformed distribution under (2) reflects the skew (asymmetry) of the underlying loss distribution.

As a combination of (1) and (2), the two-factor model (3) provides risk premium adjustments not only for the second moment, but also for higher moments, and for parameter uncertainty. We demonstrate this effect by the following numerical exhibit in Table 1.

Table 1: Transformed default frequency: an illustration

Corporate Bond	**Historical default frequency**	**Wang transform (1)**		**two-factor model (3)**	
Rating Class	**p**	**p***	**Ratio p*/p**	**p****	**Ratio p**/p**
AAA	0.00015	0.00077	5.15	0.00971	64.73
AA	0.0004	0.00185	4.62	0.01362	34.05
A	0.00075	0.00322	4.29	0.01721	22.95
BBB	0.0017	0.00659	3.87	0.02393	14.08
BB	0.0075	0.02372	3.16	0.04735	6.31
B	0.02	0.05438	2.72	0.07995	4.00
CCC	0.08	0.16977	2.12	0.18821	2.35

In Table 1, "p" represents the historical average default frequency for each bond rating class; "p*" represents the implied default under Wang transform (λ=0.45); "p**" represents the implied default under the two-factor transform (λ=0.45 and k=6)

5. Empirical Findings

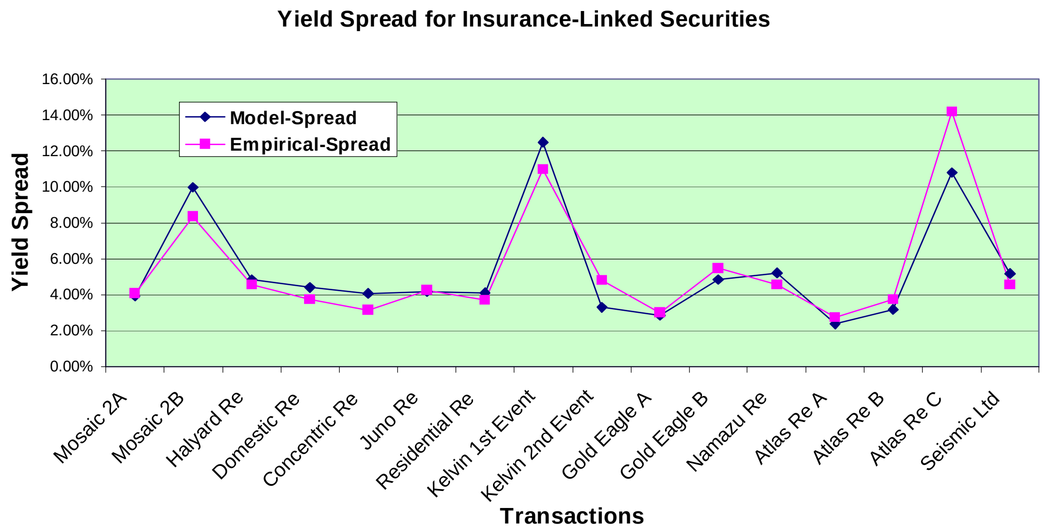
Now we present some empirical tests of the proposed pricing model (3) using observed transaction prices for cat bonds and corporate bonds. The cat bond transaction data was compiled by Lane Financial (see also Lane, 2000).

Table 2 and Figure 4 show the fitting result of the two-factor model (3) to the yields spreads for 16 cat bond transactions in 1999. Based on minimising the mean squared error, the best-fit parameters were λ=0.453, and k=5 for the student-t degrees of freedom.

Table 2. Fitted two-factor model yield spreads versus empirical yield spreads for 16 cat bond transactions in 1999

Cat bond Transaction	Probability of First $ Loss	Probability of Last $ Loss	Expected Loss given default	Model Yields Spread	Empirical Yields Spread
Mosaic 2A	0.0115	0.0012	0.3652	3.88%	4.06%
Mosaic 2B	0.0525	0.0115	0.5410	10.15%	8.36%
Halyard Re	0.0084	0.0045	0.7500	4.82%	4.56%
Domestic Re	0.0058	0.0044	0.8621	4.36%	3.74%
Concentric Re	0.0062	0.0022	0.6770	4.01%	3.14%
Juno Re	0.0060	0.0033	0.7500	4.15%	4.26%
Residential Re	0.0076	0.0026	0.5789	4.08%	3.71%
Kelvin 1st Event	0.1210	0.0050	0.3678	12.80%	10.97%
Kelvin 2nd Event	0.0156	0.0007	0.1923	3.25%	4.82%
Gold Eagle A	0.0017	0.0017	1.0000	2.81%	2.99%
Gold Eagle B	0.0078	0.0049	0.8077	4.82%	5.48%
Namazu Re	0.0100	0.0032	0.7500	5.20%	4.56%
Atlas Re A	0.0019	0.0005	0.5789	2.35%	2.74%
Atlas Re B	0.0029	0.0019	0.7931	3.15%	3.75%
Atlas Re C	0.0547	0.0190	0.5923	11.01%	14.19%
Seismic Ltd	0.0113	0.0047	0.6460	5.13%	4.56%

Figure 4. Fit the two-factor model (3) to empirical yields spreads for the 16 cat bond transaction data in 1999 compiled by Lane Financial (fitted parameters: λ=0.453 and k=5).

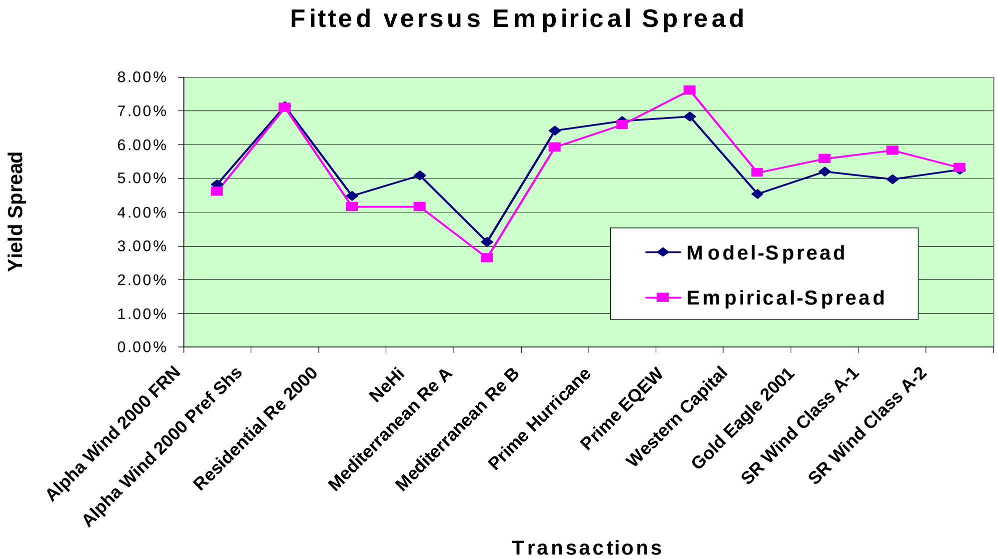


In Table 3 and Figure 5, we can see that using the fitted parameters λ=0.453, and k=5, the two-factor model (3) can explain reasonably well the 12 cat bond transactions in 2000.

Table 3. Fitted two-factor model yields spread versus empirical yields spread for 12 cat bond transactions in 2000

Cat-bond Transaction	**Probability of First $ Loss**	**Probability of Last $ Loss**	**Expected Loss given default**	**Model Yields Spread**	**Empirical Yields Spread**
Alpha Wind 2000 FRN	0.0099	0.0038	0.6364	4.82%	4.62%
Alpha Wind 2000 Pref Shs	0.0208	0.0099	0.7019	7.14%	7.10%
Residential Re 2000	0.0095	0.0031	0.5684	4.48%	4.16%
NeHi	0.0087	0.0056	0.8046	5.09%	4.16%
Mediterranean Re A	0.0028	0.0017	0.7857	3.13%	2.64%
Mediterranean Re B	0.0147	0.0093	0.7891	6.42%	5.93%
Prime Hurricane	0.0146	0.0108	0.8699	6.70%	6.59%
Prime EQEW	0.0169	0.0107	0.7870	6.84%	7.60%
Western Capital	0.0082	0.0034	0.6707	4.55%	5.17%
Gold Eagle 2001	0.0118	0.0045	0.6356	5.20%	5.58%
SR Wind Class A-1	0.0107	0.0044	0.6355	4.98%	5.83%
SR Wind Class A-2	0.0113	0.0053	0.6726	5.25%	5.32%

Figure 5. Using the 1999 fitted parameters (λ=0.453 and k=5) to test the empirical yields spreads for the 12 cat bond transaction data in 2000

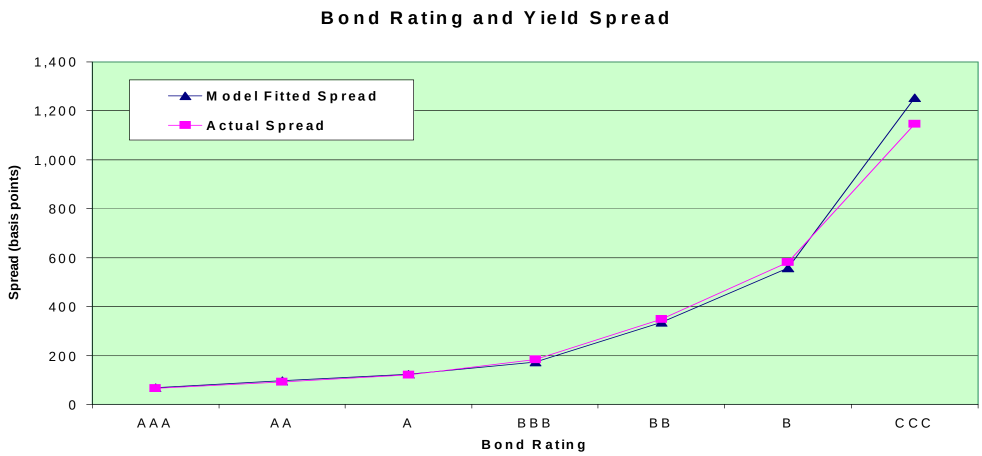


For corporate bonds, there are documented statistics on historical defaults:

- historical default probabilities by bond rating classes (e.g., Standard and Poors, Moody's), and
- the recovery rate by debt seniority (e.g., see Carty and Lieberman, 1996).

Table 1 shows historical corporate bond default frequencies for various rating classes (see also Lane, 2000). Figure 6 shows the fit of the two-factor model (3) to the grid of yields spread for corporate bonds with various credit ratings. With parameters λ=0.453, and k=6 (for the student-t degrees of freedom), we get approximately the best fit.

Figure 6. Fit the two-factor model (3) to empirical yields spreads for corporate bonds (fitted parameters: λ=0.453 and k=6).



6. Comment on the Dynamics for the Market Price of Risk

Obviously market transaction prices for cat bonds are influenced by many factors.

Firstly, cat bond prices should reflect the risk characteristics in terms of estimated default frequency and default severity. Thanks to the availability of cat modeling software, these types of information are normally available to cat bond investors.

Secondly, cat bond prices are influenced by investors' perceived attractiveness of this asset class within the broader investment environment. Over the past few years we have observed a major shift in investor's appetite for cat bonds.

Before September 11 of 2001 fund managers were less familiar (or comfortable) with the cat bond asset class. There was reluctance on the part of fund managers to expose themselves to potential career risks, since fund managers would have difficulties in explaining losses from investing in cat bonds, instead of conventional corporate bonds. At that time, because of investors' weak appetite for cat bonds, cat bonds issuers had to offer more attractive yields than corporate bonds with comparable default frequency & severity.

Over the past two years (that is, in 2002 and 2003), however, fund managers' interest in investing in the cat bond asset class has growth significantly. This was partly due to that the superior performance of the cat bond class was made known to the investment community. At the same time, the perceived credit risk of corporate bonds increased, in tandem with the general broader market. Investors began to value more the diversification benefit of low correlation between cat bond and other asset classes. As of the time of writing, there are more fund managers interested in investing in the cat bond asset class; their only complaints are not having enough cat bond issues to feed their risk appetite. Under this new environment, it has been reported that the yields spreads on cat bonds have tightened while the yields spreads on corporate bonds have widened (see Polyn's April 2003 Risk Magazine article "Seller's market for cat bonds" or Lizak's "The Prism of Cat bon, Credit, and ILW Pricing in this volume).

7. Summary

In this paper, we showed that the Wang transform directly extends the Sharpe Ratio concept to credit risks that are characterized by skewed loss distributions. With this universal pricing formula, investors can compare the risk-return trade-off of risk vehicles drawn from virtually any asset class.

The 1999 and 2000 market transaction data (as compiled by Lane Financial) indicated that cat bonds and corporate bonds offered similar risk-return trade-offs in terms of Sharpe ratio. However, cat bonds and corporate bonds showed different student degrees of freedom, k=5 and k=6, respectively. In other words, investors demanded higher risk-adjustment for parameter uncertainty for cat bonds than for corporate bonds.

Based on a recent Risk Magazine report (see Polyn, 2003), over the past two years, the relative attractiveness in yields for cat bonds and corporate bonds may have changed. Despite this fact, cat bond yields still look attractive to investors, especially given the low correlation between cat bond and the general market portfolio. Today's investors are becoming more willing to take on this new asset class.

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